AN ISOPERIMETRIC PROBLEM WITH AN INEQUALITY

CLAIRE FISHER ADLER, New York University

The problem of this paper is a generalization of the classical isoperimetric problem: "Of all curves inclosing a domain of given area, to find one of minimum length." It was suggested by Dr. R. Courant in connection with new types of variational problems, some of which can be illustrated by soap film experiments.*

1. Formulation of the problem. Let M denote a closed, continuous, sectionally smooth curve in a bounded, closed, plane region R. That is, every such curve consists of a finite number of arcs each of which has a continuously turning tangent at each of its points up to and including its end points. Such a curve can therefore have at most a finite number of corners or cusps. Also, each such curve can be represented parametrically by equations of the form: $x=x(t)$, $y=y(t)$, $t_0 \leqq t \leqq t_1$, the curve being described just once as the parameter t describes the interval t_0, t_1. The correspondence between points $P(t)$ of curve M and values of t need not however be one-to-one. That is, to an arc as P_1P_2 (Fig. 1) may correspond more than one sub interval of t_0, t_1. Such an arc will be said

FIG. 1

to be *described more than once* as $P(t)$ describes the curve once. From this fact it follows that M is not necessarily a Jordan curve. In order therefore to speak of a domain inclosed by such a curve it will be necessary to extend the ordinary concept of exterior, interior points of a (Jordan) curve. To do so, join a point $P(t)$ of curve M to any point Q not on the curve and denote by $\theta(t)$ the angle which the vector QP makes with the positive x axis. Then, as the parameter t describes the interval t_0, t_1 the vector QP generates an angle $\theta(t_1)-\theta(t_0)=2\pi n$ where the number n is independent not only of the parametric representation of the curve but also of the determination of $\theta(t)$. The number n is called the *index* of *the point Q with respect to curve M*. As is well known, when curve M is a Jordan curve, exterior, interior points are points of index 0, 1 respectively. We shall consider therefore only those curves M for which *every point not on the curve is of index* 1 *or* 0. *By definition, exterior, interior points of curve M are points of index* 0, 1 *respectively.*

Points of index 1 form an open set D consisting of a finite number of domains

*See "What Is Mathematics" by R. Courant and H. Robbins, pp. 385–397, and a paper by R. Courant to be published after the war.

inclosed by the curve. *The area of the set D is said to be the area inclosed by the curve. If D is void, M is said to inclose zero area.*

If a point P of M is a boundary point of a domain inclosed by curve M, the point P is called a *boundary point of the curve*; otherwise, point P is a *non-boundary point.*

The *length of curve M* is the sum total of the lengths of the arcs of the curve, *each arc being counted as many times as it is described.*

γ denotes the class of all such curves M.

Admissible curves are those curves of the class γ which inclose a domain or domains 1) of a prescribed fixed area $S \geqq 0$ and 2) these domains contain a given set A of fixed points in their interior or on their boundaries. Points of the set A are denoted $A_1, A_2 \cdots A_n$, $n > 2$. The problem to be solved is then:

PROBLEM. *In the class of admissible curves find one whose length is a minimum.*

The existence of a solution is assumed in Part I of the paper and proved in Part II. C always denotes a solution and $L(S)$ its length. Part I contains 1) properties of a solution C showing its form or structure and 2) continuity considerations showing the dependence of the length $L(S)$ of curve C on the area S. In addition to existence proofs, Part II contains an illustrative example showing the changing form of a solution as the area increases.

From the classical theory, it is known that, C is a circle, when the condition 2 of admissibility, called an *inequality*, is *unessential*, *i.e.*, when the problem *with* the inequality has the same solution as the problem *without* the inequality. Only the *essential* inequality is considered here.

Part I

2. Three preliminary properties of a solution C.

PROPERTY 1. *Every non-boundary point of C is on a straight line segment and every boundary point on an arc of a circle (which may be a straight line).*

a) Every non-boundary point of C is on a straight line segment since such a point must be connected to a boundary point by a curve of shortest length: *i.e.*, a *straight line segment.*

b) Every boundary point is on an arc of a circle:

Proof. From the definition of an admissible curve, C consists of a *finite* number of smooth arcs and incloses only a finite number of domains. Moreover, two domains may have a common boundary arc only if the arc is a point for otherwise removal of this common boundary arc would result in an admissible curve of smaller length than C thus contradicting the hypothesis that C is a solution. Hence *the only arcs of C which are described more than once are straight line segments.* Let G denote the finite set of boundary points consisting of:

1) *Extremities of line segments,*
2) *Points common to the boundaries of different domains,*
3) *Points of the set A which are on C.*

Then, if a boundary point P is not a point of the set G, there exists a positive quantity ϵ such that the circle whose center is P and radius ϵ meets C in only two points, R and Q (Fig. 2), contains points of one domain only and does not contain any points of the set A. From the classical theory, it is known that in this circular domain the arc of smallest length joining R and Q is an arc of a circle. Hence the property is proved for a boundary point not belonging to the set G.

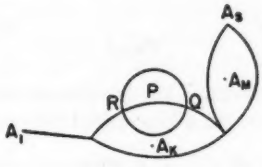


FIG. 2

However, since these arguments are valid as long as the point P is not a point of the set G, an extremity of an arc of a circle must be a point of the set G. C therefore consists of straight line segments and arcs of circles. If extremities of arcs of circles be called *end-points* it has also been shown that:

COROLLARY: *End-points of C belong to the set G.*

PROPERTY 2. *If one arc of a circle of C is convex (concave), every arc of a circle is convex (concave).*

For, otherwise, it is always possible to replace a convex and a concave arc by their respective chords KM and RS (Fig. 3a) where the arcs are so chosen that the resulting curve is still admissible. But since chords have smaller length than their arcs, the new curve (Fig. 3b) has smaller length than C and from this contradiction Property 2 follows.

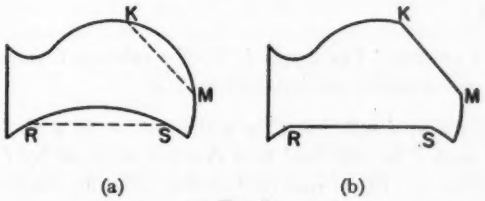
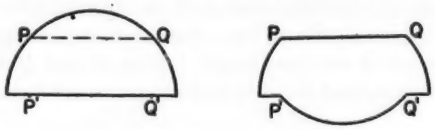


FIG. 3

PROPERTY 3. *Arcs of circles of C have equal radii.*

For, otherwise, there exist chords PQ of radius r_1 and $P'Q'$ of radius $r_2 \neq r_1$ such that the chords are equal and the domains inclosed by these arcs and their respective chords do not contain any points of the set G or of the set A. Then the curve which results by interchanging arc PQ with arc $P'Q'$ is still an admis-

sible curve but an end point such as P' (Fig. 4b) does not belong to the set G. Since this contradicts corollary, property 1, the property is proved.



(a) (b)

FIG. 4

3. Continuity considerations. The dependence of the length $L(S)$ of C on the area S is given by a theorem whose proof depends on the following:

LEMMA. *The length $L(S)$ of C is a) a monotonic increasing function of S if an arc of a circle of C is convex and b) a monotonic decreasing function of S if the arc is concave.*

Proof: If a convex, or a concave arc of a circle of C, inclosing with its chord a domain not containing any fixed points, be replaced by its chord, the new curve E 1) is still admissible, and 2) incloses area $S'<S$ if the arc is convex and area $S'>S$ if the arc is concave. Moreover, since a chord is less than its subtended arc, the length $E(S')$ satisfies:

$$E(S') < L(S). \tag{1}$$

But, since $L(S')$ is the length of a solution inclosing area S',

$$E(S') > L(S'). \tag{2}$$

Therefore:

$$L(S') < L(S) \tag{3}$$

where $S'<S$ if the arc is convex and $S'>S$ if the arc is concave and thus the lemma is proved.

CONTINUITY THEOREM. *The length $L(S)$ of a solution C inclosing a domain or domains of total area S varies continuously with S.*

Proof. Consider first a solution C in which an arc of a circle of C is convex. Let a domain of area h be adjoined to a domain inclosed by C by replacing an arc PQ by a circular arc PQ of different radius (Fig. 5). Such a domain can always be chosen so that the resulting curve E is still admissible when the inclosed area is $S+h$. Moreover, if h is a sufficiently small positive number there exists an $\epsilon>0$ such that if $E(S+h)$ denote the length of E, we have

$$0 < E(S+h) - L(S) < \epsilon \tag{1}$$

where $L(S)$ denotes the length of C. But,

$$E(S+h) \geqq L(S+h) \tag{2}$$

since $L(S+h)$ is the length of a solution inclosing area $S+h$. From the lemma just proved:

$$L(S + h) > L(S). \tag{3}$$

Therefore

$$0 < L(S + h) - L(S) < \epsilon \tag{4}$$

and the theorem is proved when an arc of C is convex. If an arc of C is concave, the domain of area h may be chosen 1) as a sub-domain of a domain inclosed by C and 2) so that the domain contains no points of the set A. Then E incloses area $S-h$; and if $S-h$ replace $S+h$, the above inequalities still hold. That is, the theorem is proved also when an arc of C is concave. Since these arguments hold for any value of S however small, $L(S)$ is continuous at $S=0$.

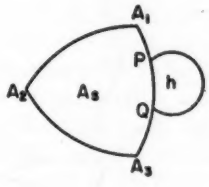


FIG. 5

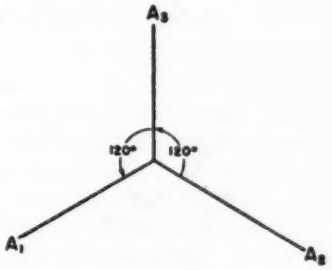


FIG. 6

4. Additional properties of C. Steiner's classic problem "To find a point O at a minimum distance from three fixed points," is the problem of this paper when $S=0$ and $A_1A_2A_3$ are the given points. Its solution in theorem form is:

THEOREM A. (STEINER). *The point O at a minimum distance from three fixed points $A_1A_2A_3$ is the point O (Fig. 6) from which each of the three sides of the triangle $A_1A_2A_3$ subtends an angle of 120°, if no angle of the triangle is greater than or equal to 120°; otherwise, it is the vertex of the largest angle of the triangle.*

PROPERTY 4. *When $S=0$, C consists of straight line segments and no more than three of them meet in a point.*

For, by definition, every point of C is a non-boundary point when $S=0$. Hence, by property 1, C consists of straight line segments. Suppose now that more than three of these segments meet in a point P; then one of the angles at P say A_1PA_3 is less than 120°, and since C is a solution this is the shortest path connecting points A_1, P, and A_3. Since this contradicts Theorem A, Property 4 follows.

Let a point of meeting of exactly 3(2) line segments of C be called a *regular* (*degenerate*) contact point, or, where no confusion will arise, simply a *contact* point.

From Theorem A it then follows immediately that:

PROPERTY 5. *a) Each obtuse angle formed by lines meeting at a regular contact point equals* 120°; *b) the obtuse angle formed at a degenerate contact point is* $\geqq 120°$.

The smoothing effect of the minimizing property is shown by:

PROPERTY 6. *A line segment of C is tangent to an arc of a circle of C at its point of meeting with the arc.*

Proof. Suppose the contrary, *i.e.*, a solution C has a corner at an end point e lying on a line segment of C. Fig. 7a. There is then a point P of C so near e that if r is the radius of arcs of circles of C, a circle of radius r and through P meets C

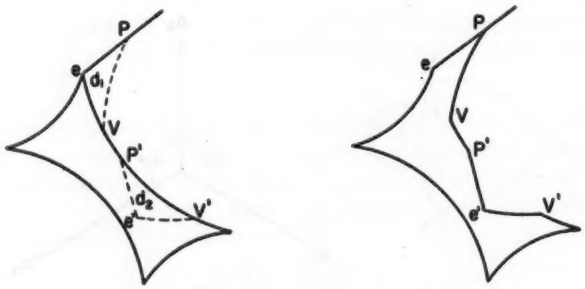


FIG. 7

in another point V and incloses with arc PeV a domain d_1 of arbitrarily small area and outside any domain inclosed by C. Also, there exists a closed domain d_2 congruent to d_1 and its boundary, and not containing any points of the set A. Now, replace arcs PeV and $P'V'$ by arcs PV and $P'e'V'$ respectively. There results an admissible curve E, Fig. 7b (by hypothesis, the arc Pe was described *twice* and removed *once*). Moreover E has the same length as C, for E is simply curve C except for 1) arc PeV which is replaced by arc PV and 2) arc $P'V'$ which is replaced by arc $P'e'V'$, and $P'e'V'$ is congruent to PeV. But E has an endpoint as P' not belonging to the set G (page 60) and hence by corollary, property 1, its length and therefore the length of C is not a minimum. From this contradiction the property follows.

Similar arguments hold if 1) e is a point common to the boundary of two different domains or 2) C contains both a contact point and incloses a domain of area $S>0$. Hence, if a contact point be called a domain of zero area:

COROLLARY 1. *If any domain inclosed by C has area greater than (equal to) zero, so has every domain.*

COROLLARY 2. *C has no corners when* $S>0$ *except possibly boundary points belonging to the set A.*

COROLLARY 3. *If C contains a line segment when $S>0$, arcs of circles of C are concave. For, otherwise, a point of meeting of a convex arc and a straight line segment would be a corner.*

LEMMA. *If S is sufficiently small, no domain contains more than one fixed point, and arcs of circles of C are concave.*

Proof. The maximum diameter of any domain is less than or equal to twice the radius r of arcs of circles of C so if d is the minimum distance between any two fixed points of the set A, and if $2r<d$, no domain contains more than one fixed point. For such a value of r, a domain must be joined to a fixed point by a line segment (of length greater than zero), and by corollary 3, property 6, arcs of circles are concave. Hence the lemma is proved.

PROPERTY 7. *C incloses at most $n-2$ domains.*

Proof. For simplicity, let a point common to the boundary of two domains be called a line segment joining the same. Similarly, if an end-point of a domain is a fixed point, the two points are said to be joined by a line segment of length zero. From corollary 1, property 6, it follows that if any domain inclosed by a solution C has area $S>0$, so has every domain. Suppose first that area $S>0$ and the conditions of the lemma are satisfied. Then each domain has at least three end-points of which, if k is the number of domains inclosed, $2(k-1)$ of them are on the $k-1$ line segments connecting these k domains. But each of the remaining $3k-2(k-1)=k+2$ end-points must be joined to a fixed point by a line segment and this line segment as well as those joining domains may contain more than one point of the set A. Thus the correspondence between these $k+2$ end-points and the n points of the set A is: 1) To each end-point e_i, $i=1, 2, \cdots, k+2$, corresponds a point A_j (or points), $j=1, 2, \cdots, n$, and 2) no two end-points correspond to the same fixed point. Therefore $k+2\leqq n$ or $k\leqq n-2$, where k is the number of domains inclosed by C, and property 7 is proved when area $S>0$.

If area $S=0$, by corollary 1, property 6, every domain d_i has zero area and d_i is a contact point. Hence there are again three line segments corresponding to each domain and the remaining arguments are the same as those just given.

If a domain with three and only three end-points be called a *triangular domain*, the next property is:

PROPERTY 8. *When 1) the area inclosed by C is sufficiently small and 2) the set A contains more than three points, C incloses more than one domain and each domain is triangular.*

Proof. Let K denote the class of admissible curves each of which incloses zero area and has more than three line segments meeting at a point. Consider the set of numbers representing the difference between the length $L(0)$ of a solution inclosing zero area and the length $M(0)$ of a curve M of class K. This set of positive numbers has a greatest lower bound $B>0$ since otherwise there would be a curve of the set whose length would be arbitrarily near $L(0)$ and by Theorem A this is impossible. Hence

$$M(0) - L(0) \geqq B. \tag{1}$$

Suppose now that C incloses an area satisfying lemma, page 65. Then in each domain d_i, $(1 \leqq i \leqq n-2)$, inclosed by C, select a point P_i (where P_i is the fixed point if d_i contains one) and join P_i to each end-point of d_i by a line segment. *Remove all arcs of circles of* C. There results a curve M of class K, if any domain of C has *more than three end-points*. Suppose such is the case. If then $\epsilon > 0$ is the area inclosed by C, there exists a positive quantity δ depending on ϵ such that the length $L(\epsilon)$ of C satisfies

$$|L(\epsilon) - M(0)| = \delta, \tag{2}$$

where if ϵ is small so is δ.

Suppose now that δ is chosen so small that

$$\delta < B. \tag{3}$$

Then from 1), 2), and 3) it follows that

$$L(\epsilon) > L(0), \tag{4}$$

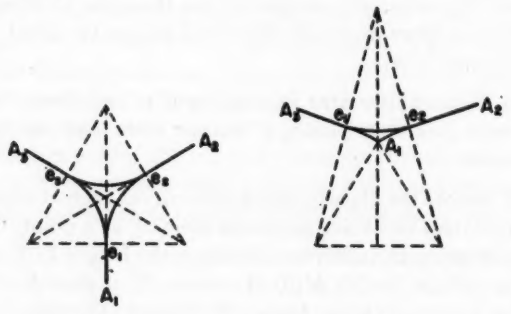
which contradicts lemma, page 62, and hence property 8 is proved, and also:

COROLLARY 1. *When the area inclosed by C is sufficiently small, C incloses* $n-2$ *domains.*

COROLLARY 2. *A solution curve inclosing area near zero is near a solution curve inclosing zero area.*

Two additional properties, whose proofs are omitted because of their extremely elementary nature, are:

PROPERTY 9. *If each end-point of a triangular domain of C is a point of tangency of the arcs through it, the end-points form an equilateral triangle* (*Fig.* 8*a*); *if two and only two end-points are points of tangency, the end-points form an isosceles triangle* (*Fig.* 8*b*).



(a) (b)

FIG. 8

PROPERTY 10. *If a domain inclosed by C has four and only four end-points each of which is a point of tangency of arcs through it, these end-points form a rectangle* (*Fig.* 9).

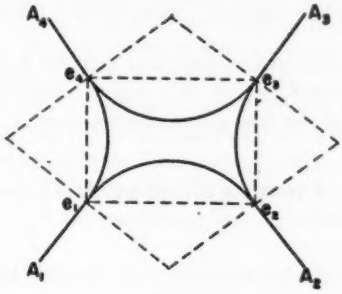


FIG. 9

This completes the description of the structure of C.

Part II

5. Existence Proof. By definition, all admissible curves inclose the prescribed constant area.

When $S>0$, denote by α the sub-class of admissible curves having properties 1, 2, 3 and 7 and

When $S=0$, denote by β the sub-class of admissible curves having properties 1, 4 and 7.

From the arguments in part I, if a solution exists in the class $\alpha(\beta)$ it is the solution of the original problem. That this solution exists is shown as follows:

EXISTENCE THEOREM. *There exists a solution in* α *and a solution in* β.

Proof. Let k denote the number of end-points of a curve of α or contact points of a curve of β. Then, from property 7, there exists a positive integer s such that

$$0 < k < s. \tag{1}$$

It will be shown first that a solution exists in α. Let r denote the radius of arcs of circles of a curve E of α, x_i, $y_i (i=1, 2, \cdots, k)$ the rectangular coordinates of end-points P_i of E, and

$\lambda = \arctan r$. Then if $E(S)$ denote the length of E,

$$E(S) = \Phi(x_1, \cdots, x_k, y_1, \cdots, y_k, \lambda) \tag{2}$$

where the variables $x_1, \cdots, x_k, y_1, \cdots, y_k, \lambda$, denoted simply x, y, λ, satisfy the subsidiary condition:

$$f(x, y, \lambda) = S, \tag{3}$$

and both Φ and f are continuous functions of x, y, λ. Since all of these variables

lie in a bounded, closed region T, by elementary function theory $E(S)$ has a minimum in T. Let $L_k(S)$ denote this minimum. Now from 1), k can take on only a finite set of values. The corresponding values assumed by $L_k(S)$ therefore are finite in number. In this finite set there exists a minimum $L(S)$. C is a curve whose length is $L(S)$.

If x_i, y_i, are coordinates of end-points of a curve E of β and $E(0)$ its length, then $E(0)$ is a continuous function of x and y alone and the remaining arguments are the same as those just given. Hence there exists a solution in β and the theorem is proved.

6. Example. The set A consists of four points A_i, $(i=1-4.)$ In a rectangular coordinate system in which a_i, b_i are coordinates of A_i, $a_1=b_1=0$; $a_1<a_4<a_2<a_3$; $b_2<b_1<b_3<b_4$, Fig. 10.

A solution curve C inclosing zero area is shown in Fig. 10 where C consists of the line segments containing the points A_i and meeting at contact points O_1O_2. Another solution curve C' (dotted line, Fig. 10) has its contact points $O_1'O_2'$ ly-

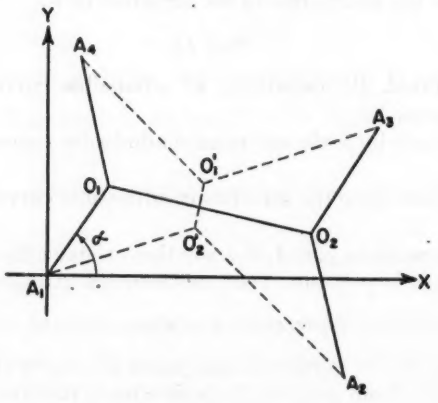


FIG. 10

ing on arcs 1) whose chords are A_4A_3 and A_1A_3 and 2) in which a 120° angle may be inscribed. If α is the inclination of A_1O_1 ,then

$$\tan\alpha=\frac{b_3-(a_4-a_3)\sin 60^\circ+(b_4-b_3)\cos 60^\circ}{a_3+(a_4-a_3)\cos 60^\circ+(b_4-b_3)\sin 60^\circ}.$$

The changing form of a solution as the area S increases is shown graphically in Fig. 11 and Fig. 12.

Fig. 11 shows a solution curve inclosing area S near zero. The number of domains inclosed is two (property 8, corollary 1), and the end-points of each domain form an equilateral triangle (property 9). Boundaries of domains are concave arcs of circles of equal radii (properties 2, 3, 6, corollary 3).

Domains are connected by line segments and fixed points are joined to domains by line segments (property 1).

As S increases, the line segment joining domains shrinks and finally disappears. A solution curve C then incloses only one domain, Fig. 12, and its end-points form a rectangle (property 10) as long as every end-point is a point of tangency of the arcs through it.

For increasing values of S, line segments through the fixed points become shorter and finally disappear and when arcs of circles of a solution curve C are straight lines (*i.e.*, circles of infinite radius) C is the quadrilateral $A_1A_2A_3A_4$, and S is its area (dotted line, Fig. 12).

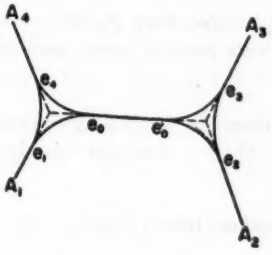


FIG. 11

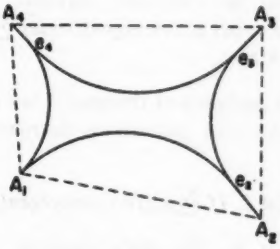


FIG. 12

For all larger values of S, curve C consists of 4 convex arcs of circles, and every end-point is a fixed point as long as S is not too large. But when the area S is so large that arcs through the vertex of the second largest angle of the quadrilateral have a common tangent, a solution C is a circle having the vertex of the largest angle of the quadrilateral on its interior. This point then represents an *unessential* inequality.

7. Generalizations. Only a few additional arguments are needed to solve the problem when any fixed point is replaced by an admissible curve.

The theory cannot be extended to a space of three dimensions, however, if by a three-dimensional space generalization is meant one in which the area is replaced by volume and the length by surface. This is seen most easily by considering the surface of minimum area inclosing a given volume, *i.e.*, a sphere. When a sphere is joined to the fixed points by spine-like projections, the area of the resulting surface can be made to approach the area of the surface of the sphere to any degree of accuracy, but in no case are the two areas equal. The problem leads directly to the theory of minimal surfaces, however, if the surface is required to contain not only the given points but also the line segments joining them. It is known that a surface inclosing a given volume has least area when its mean curvature is constant, but a general treatment of the problem along the lines of this paper has not yet been given.

CONVERGENT MONOTONE SERIES

R. W. HAMMING, University of Louisville

1. Introduction. We shall use the words "monotone series" to mean a series $\sum a_n$ with positive monotone decreasing terms, $a_1 \geqq a_2 \geqq \cdots \geqq 0$. The following theorems form a background for the present paper and their proofs are so well known that they need not be repeated here.*

THEOREM 1. *If $\sum a_n$ is a convergent series with positive terms, then $a_n \to 0$.*

This is a "best possible" theorem in the following sense.

THEOREM 1*a*. *For every sequence (p_n) approaching infinity, no matter how slowly, there exists a convergent series $\sum a_n$ with positive terms, such that $p_n a_n$ does not approach zero.*

If the hypotheses of theorem 1 are increased by supposing that the terms are both positive and monotone decreasing, then a stronger conclusion can be drawn.

THEOREM 2. *If $\sum a_n$ is a convergent monotone series, then $na_n \to 0$.*

THEOREM 2*a*. *For every sequence (p_n) approaching infinity, no matter how slowly, there exists a convergent monotone series $\sum a_n$ such that $np_n a_n$ does not approach zero.*

This paper extends the above sequence of theorems and for this the Cauchy Condensation Test is needed.

THEOREM 3. (*Cauchy Condensation Test*) *If $\sum a_n$ is a convergent (divergent) monotone series, then the series $\sum 2^n a_{2^n}$ is convergent (divergent).*

2. Cauchy derived series. The Cauchy Condensation Test replaces a monotone series $\sum a_n$ by a derived series $\sum 2^n a_{2^n}$, which we shall call *the first Cauchy derived series*. The derived series of the first Cauchy derived series will be called the *second Cauchy derived series, etc.* As shown in the following example, the first Cauchy derived series need not be monotone. Let $\sum a_n$ be any convergent monotone series. From it obtain a series $\sum b_n$ according to the following rule: for n satisfying $2^{2k} < n \leqq 2^{2k+2}$ set $b_n = a_{2^{2k+2}}$. This makes $\sum b_n$ both monotone and convergent. However its first Cauchy derived series is not monotone since

$$2^{2k+1} b_{2^{2k+1}} < 2^{2k+2} b_{2^{2k+2}}.$$

THEOREM 4. *If $\sum a_n$ is a convergent monotone series and the first Cauchy derived series is also monotone, then $n \log n a_n \to 0$.*

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* See, for example, any edition of Knopp, K. Theorie und Anwendung der Unendlichen Reihen.

Proof. The first Cauchy derived series satisfies theorem 2 so that

$$n2^n a_{2^n} \to 0,$$

$$(\log 2) n 2^n a_{2^n} \to 0,$$

$$2^n \log 2^n a_{2^n} \to 0. \tag{2.1}$$

Thus a sub-sequence of $(n \log na_n)$ approaches zero. Suppose, contrary to what we wish to prove, that $(n \log na_n)$ does not approach zero. Then there exists an $\epsilon > 0$ and infinitely many m_i such that

$$m_i \log m_i a_{m_i} \geqq \epsilon.$$

However, by (2.1) there exists an n_0 such that for all $n \geqq n_0$

$$2^n \log 2^n a_{2^n} \leqq \epsilon/8.$$

Thus for $m_i \geqq 2^{n_0}$ and satisfying $2^n \leqq m_i < 2^{n+1}$

$$\epsilon \leqq m_i \log m_i a_{m_i} < 2^{n+1} \log 2^{n+1} a_{2^n}$$

$$< 2\left(\frac{n+1}{n}\right) 2^n \log 2^n a_{2^n} \leqq 4 \cdot \epsilon/8 = \epsilon/2$$

which is a contradiction. Thus $(n \log na_n)$ approaches zero.

It is often thought, contrary to theorem 2*a*, that if $\sum a_n$ is merely monotone and convergent then necessarily $n \log na_n$ approaches zero.* It is true that *if* $\sum a_n$ *is a convergent monotone series and if* $n \log na_n$ *approaches a limit, then this limit is zero,*† but the following elementary example‡ shows that $n \log na_n$ need not approach a limit. Define the terms of $\sum a_n$ as $a_n = 1/(k^{k^2} \log k^{k^2})$ whenever n satisfies $(k-1)^{(k-1)^2} < n \leqq k^{k^2}$. Clearly $\sum a_n$ is monotone. The proof of its convergence runs as follows: for every N there exists a k such that $N < k^{k^2}$, and hence

$$S_N = \sum_{n=2}^{N} a_n \leqq \sum_{n=2}^{k^{k^2}} a_n = \sum_{m=2}^{k} \frac{m^{m^2} - (m-1)^{(m-1)^2}}{m^{m^2} \log m^{m^2}}$$

$$\leqq \sum_{m=2}^{\infty} \frac{1}{\log m^{m^2}} = \sum_{m=2}^{\infty} \frac{1}{m^2 \log m} = C.$$

Thus S_N is bounded and must approach a limit. However, it is clear that for $n = k^{k^2}$

$$n \log na_n = 1.$$

THEOREM 4*a*. *For every sequence* (p_n) *approaching infinity, no matter how slowly, there exists a convergent monotone series* $\sum a_n$ *with a monotone first Cauchy derived series such that* $n \log np_n a_n$ *does not approach zero.*

* For references see Pringsheim, A. Allgemeine Theorie der Divergenz und Convergenz von Reihen mit positiven Gliedern, Math. Annalen, V. 35, 1890, pp. 297–394.

† The proof follows readily from the contrary assumption.

‡ "Elementary" with respect to those in the literature.

Proof. Applying theorem 2a to the sequence (p_{2^n}) there exists a convergent monotone series $\sum b_n$ such that $nb_np_{2^n}$ does not approach zero. Set $b_n=2^na_{2^n}$ and multiply by log 2:

$$n2^na_{2^n}p_{2^n}\log 2 = 2^n \log 2^np_{2^n}a_{2^n}.$$

Define a_m for m satisfying $2^{n-1}<m\leqq 2^n$ as equal to a_{2^n}. Thus $\sum a_n$ is monotone with a monotone first Cauchy derived series. The proof of the convergence runs as follows:

$$\sum a_n = \sum (2^n - 2^{n-1})a_{2^n} = \tfrac{1}{2}\sum 2^na_{2^n} = \tfrac{1}{2}\sum b_n.$$

Since a sub-sequence of $(n \log np_na_n)$ does not approach zero the sequence itself cannot approach zero, and we have proved the theorem.

COROLLARY TO THEOREM 4. *If $\sum a_n$ is a convergent monotone series and the first k Cauchy derived series are monotone, then $n \log n \log_2 n \log_3 n \cdots \log_k na_n \to 0$, where $\log_2 n \equiv \log\log n$, $\log_3 n \equiv \log\log\log n$, etc.*

Proof. We have actually to prove an infinite sequence of theorems, one for each integer k. To prove these we apply mathematical induction. Theorem 4 provides a basis for the induction with $k=1$.* Thus assume that the corollary is true for $k-1$. Label the m-th Cauchy derived series as $\sum a_n^{(m)}$. We have by the hypotheses of the corollary that the $k-1$th Cauchy derived series of $\sum a_n^{(1)}$ is monotone, so that by the hypotheses of the induction,

$$n \log n \log_2 n \cdots \log_{k-1} na_n^{(1)} \to 0.$$

But $a_n^{(1)}=2^na_{2^n}$, and the above becomes,

$$2^n n \log n \log_2 n \cdots \log_{k-1} na_{2^n} \to 0.$$

Multiply by the constant $\log 2 \log_2 2 \cdots \log_k 2$:

$$2^n \log 2^n \log_2 2^n \cdots \log_k 2^na_{2^n} \to 0.$$

Thus a sub-sequence of $(n \log n \log_2 n \cdots \log_k na_n)$ approaches zero. The rest of the proof follows that of theorem 4.

COROLLARY TO THEOREM 4a. *For every sequence (p_n) approaching infinity, no matter how slowly, there exists a convergent monotone series $\sum a_n$ with the first k Cauchy derived series also monotone such that $n \log n \log_2 n \cdots \log_k np_na_n$ does not approach zero.*

The proof is left to the reader.

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* Actually we could have used theorem 2 as a basis and performed the induction then and not developed theorem 4.

MAXIMUM ANGULAR VARIATION UNDER SMALL DISPLACEMENTS

T. Y. THOMAS, Indiana University

1. Angular variation under small displacements. Let x^α and $\bar{x}^\alpha (\alpha = 1, 2, 3)$ denote coordinates of points relative to the same rectangular system. A displacement of space or of a material body referred to this system can be represented by $\bar{x}^\alpha = x^\alpha + u^\alpha(x)$ where the $u^\alpha(x)$ are the components of a vector. We assume the u^α to be continuous and to have continuous first partial derivatives. To this requirement we add the further condition that the u's and their derivatives are small in the sense that the squares and higher powers of these quantities can be neglected. This is the usual assumption in the theory of elastic displacements.

If ν_1 and ν_2 are unit vectors at a point P, and θ is the angle determined by these vectors, then the change $\delta\theta$ produced in the angle θ as a result of the above displacement is given by*

$$\sin\theta\delta\theta = (e_1 + e_2)\cos\theta - 2e_{\alpha\beta}\nu_1^\alpha\nu_2^\beta, \tag{1}$$

where

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$$

and

$$e_1 = e_{\alpha\beta}\nu_1^\alpha\nu_1^\beta, \qquad e_2 = e_{\alpha\beta}\nu_2^\alpha\nu_2^\beta.$$

Observe that in the formula for $e_{\alpha\beta}$ we have written u_α instead of u^α for the "sake of appearances" and this is legitimate since we shall limit ourselves to rectangular coordinate systems. The "comma" denotes partial differentiation, and of course the summation convention is employed. If $\nu_1 = \nu_2$ or $\nu_1 = -\nu_2$ we have $\theta = 0$ or $\theta = \pi$ and for each of these cases $\delta\theta = 0$. We exclude these two cases and thus suppose $0 < \theta < \pi$ in the following discussion.

2. An existence proof. The actual proof of the existence of a maximum or minimum value of $\delta\theta$ is immediate. In fact each of the directions ν_1 and ν_2 can be interpreted as a point on a unit sphere and the product of these spheres is a bicompact topological space. Since $\delta\theta$ is obviously a continuous function over this space, it follows that $\delta\theta$ assumes maximum and minimum values at points of the space.† With the existence of directions giving maximum and minimum values of $\delta\theta$ thus assured, we proceed to the actual determination of these directions and to the calculation of the maximum and minimum values of $\delta\theta$ involved.

3. Stationary values of the angular variation. A first insight into the problem of finding the above directions ν_1 and ν_2 is given by the following fact. *Stationary values of $\delta\theta$ can occur only for $\theta = \pi/2$ unless $\delta\theta = 0$ for arbitrary directions at P.* To prove this we choose rectangular axes with origin at P and such that the vectors ν_1 and ν_2 lie in the x^1, x^2 plane. Let ν_1 and ν_2 make angles θ_1 and θ_2 with the x^1 axis and suppose furthermore that the axes are so taken that $\theta = \theta_2 - \theta_1 > 0$. Then

* *Cf.* Love, The Mathematical Theory of Elasticity, Cambridge, 4th ed., 1927, p. 62. Also S. Timoshenko, Theory of Elasticity, McGraw-Hill, 1934, p. 192.

† See P. Alexandroff and H. Hopf, Topologie, Berlin, J. Springer, 1935, p. 96.

$$\nu_1^1 = \cos\theta_1, \qquad \nu_1^2 = \sin\theta_1, \qquad \nu_1^3 = 0,$$
$$\nu_2^1 = \cos\theta_2, \qquad \nu_2^2 = \sin\theta_2, \qquad \nu_2^3 = 0.$$

Substituting these values of the ν's and θ into formula (1) for $\delta\theta$ we have

$$\begin{aligned}\sin(\theta_2 - \theta_1)\delta\theta = {} & [e_{11}\cos^2\theta_1 + 2e_{12}\sin\theta_1\cos\theta_2 + e_{22}\sin^2\theta_1 + e_{11}\cos^2\theta_2 \\ & + 2e_{12}\sin\theta_2\cos\theta_2 + e_{22}\sin^2\theta_2]\cos(\theta_2 - \theta_1) \\ & - 2[e_{11}\cos\theta_1\cos\theta_2 + e_{12}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ & + e_{22}\sin\theta_1\sin\theta_2].\end{aligned} \tag{2}$$

If the vectors ν_1 and ν_2 yield a stationary value of $\delta\theta$ we must have $\delta\partial\theta/\partial\theta_1 = 0$. After performing the differentiation involved in this condition, let us set $\theta_1 = 0$ whereupon $\theta_2 = \theta$. This is permissible since we can suppose that the coordinate system has been chosen to fulfill this condition (in addition to the above requirements). We thus obtain, after some trigonometric simplification:

$$\sin^2\theta\,\frac{\partial\delta\theta}{\partial\theta_1} = e_{11} - e_{22}.$$

Hence the above condition for stationary $\delta\theta$ gives $e_{11} = e_{22}$. Substitution of this relation in (2) gives

$$\delta\theta = -2e_{12}\sin^2\theta. \tag{3}$$

Similarly the condition $\partial\delta\theta/\partial\theta_2 = 0$ in conjunction with the relation $e_{11} = e_{22}$ yields, after some calculation,

$$\frac{\partial\delta\theta}{\partial\theta_2} = -2e_{12}\sin\theta\cos\theta = 0. \tag{4}$$

Now if $e_{12} = 0$ (in the special coordinate system selected) then from (3) we have $\delta\theta = 0$. Since our consideration applies in particular to directions ν_1 and ν_2 giving an absolute maximum or minimum variation $\delta\theta$ this result shows that there is no variation in angle for arbitrary directions at P. If, however, $e_{12} \neq 0$ it follows from (4) that $\cos\theta = 0$ and hence $\theta = \pi/2$ (the case $\sin\theta = 0$ is excluded by the assumption $0 < \theta < \pi/2$). This completes the proof of the italicized statement.

4. Formal simplification of the main problem. The result just proved suggests that we replace (1) by the simplified equation which is obtained by taking $\theta = \pi/2$. This gives

$$\delta\theta = -2e_{\alpha\beta}\nu_1^\alpha\nu_2^\beta, \tag{5}$$

where

$$\sum \nu_1^\alpha\nu_1^\alpha = 1, \qquad \sum \nu_2^\alpha\nu_2^\alpha = 1, \qquad \sum \nu_1^\alpha\nu_2^\alpha = 0.$$

The first two of these latter equations express the fact that ν_1 and ν_2 are unit vectors, which is inherent in (1), while the last equation contains the require-

ment that the vectors are perpendicular. We now seek stationary values of $\delta\theta$ in the set of all values of $\delta\theta$ determined by perpendicular vectors ν_1 and ν_2. For the determination of the perpendicular directions ν_1 and ν_2 which give these stationary values of $\delta\theta$ we take a rectangular coordinate system with origin at P relative to which $\epsilon_{\alpha\beta}=\epsilon_\alpha\delta_{\alpha\beta}$. We suppose furthermore that the axes of this system are so enumerated that $\epsilon_1\geqq\epsilon_2\geqq\epsilon_3$. The above equation (5) now becomes

$$\delta\theta = -2\sum \epsilon_\alpha \nu_1^\alpha \nu_2^\alpha, \tag{6}$$

and the perpendicular directions ν_1 and ν_2 are those which satisfy the condition

$$\sum \epsilon_\alpha(\nu_2^\alpha\delta\nu_1^\alpha + \nu_1^\alpha\delta\nu_2^\alpha) = 0, \tag{7}$$

in which the variations $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$ are subject to the conditions

$$\sum \nu_1^\alpha\delta\nu_1^\alpha = 0, \qquad \sum \nu_2^\alpha\delta\nu_2^\alpha = 0, \qquad \sum (\nu_2^\alpha\delta\nu_1^\alpha + \nu_1^\alpha\delta\nu_2^\alpha) = 0. \tag{8}$$

To find the vectors ν_1 and ν_2 satisfying (7) we multiply the three equations (8) by constants k_1, k_2 and k_3 and add the resulting equations to (7) in accordance with the usual procedure. This gives

$$\sum [k_1\nu_1^\alpha + (\epsilon_\alpha + k_3)\nu_2^\alpha]\delta\nu_1^\alpha + \sum [(\epsilon_\alpha + k_3)\nu_1^\alpha + k_2\nu_2^\alpha]\delta\nu_2^\alpha = 0.$$

Equating the coefficients of $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$ to zero we now have

$$k_1\nu_1^\alpha + (\epsilon_\alpha + k_3)\nu_2^\alpha = 0, \tag{9}$$

$$(\epsilon_\alpha + k_3)\nu_1^\alpha + k_2\nu_2^\alpha = 0, \tag{10}$$

and the solutions ν_1 and ν_2 of these equations (for suitable selections of the constants k_1, k_2 and k_3) give the required directions. Now deduce from (9) and (10) those equations which result by elimination of ν_1^α and ν_2^α in turn. They are

$$[(\epsilon_\alpha + k_3)^2 - k_1k_2]\nu_1^\alpha = 0, \qquad [(\epsilon_\alpha + k_3)^2 - k_1k_2]\nu_2^\alpha = 0 \tag{11}$$

in which there is of course no summation on the index α. If none of the bracket expressions in (11) vanish, the only solution of these equations is given by $\nu_1=\nu_2=0$. If only one of the bracket expressions is equal to zero, for example that corresponding to $\alpha=1$, then the vectors ν_1 and ν_2 will lie along the x^1 axis. This gives $\theta=0$ or $\theta=\pi$ according as ν_1 and ν_2 have the same or the opposite direction on the x^1 axis and this case has been excluded from the discussion. Hence *at least two of the bracket expressions in* (11) *must be equal to zero.* A slight simplification in (9), (10) and (11) is obtained as follows. Multiplying (9) and (10) respectively by ν_1^α and ν_2^α and summing on the repeated index α we see that $k_1=k_2=k$. Hence these equations become

$$k\nu_1^\alpha + (\epsilon_\alpha + k_3)\nu_2^\alpha = 0, \qquad (\epsilon_\alpha + k_3)\nu_1^\alpha + k\nu_2^\alpha = 0, \tag{12}$$

$$[(\epsilon_\alpha + k_3)^2 - k^2]\nu_1^\alpha = 0, \qquad [(\epsilon_\alpha + k_3)^2 - k^2]\nu_2^\alpha = 0. \tag{13}$$

We have shown that all pairs of perpendicular directions ν_1 and ν_2 which

yield stationary values of $\delta\theta$ in the set of values of $\delta\theta$ determined by perpendicular directions are given as solutions of (12). We now show that any pair of perpendicular directions ν_1 and ν_2 satisfying (12) will likewise give a stationary value of $\delta\theta$ without the restriction that variations in direction satisfy the condition of perpendicularity, *i.e.*, the third equation (8). Removing this latter condition we must now determine the variation $\delta(\delta\theta)$ of $\delta\theta$ directly from (1) in which $\cos\theta = \Sigma\nu_1^\alpha\nu_2^\alpha$ in terms of the unit vectors ν_1 and ν_2. Making the required variations and then imposing the condition that the vectors ν_1 and ν_2 are perpendicular initially, we are led to the condition

$$\delta(\delta\theta) = [(e_1 + e_2)\delta_{\alpha\beta} - 2e_{\alpha\beta}](\nu_2^\beta\delta\nu_1^\alpha + \nu_1^\alpha\delta\nu_2^\beta) = 0.$$

Or, making the substitution $e_{\alpha\beta} = \epsilon_\alpha\delta_{\alpha\beta}$ we have

$$\sum \tau_\alpha\nu_2^\alpha\delta\nu_1^\alpha + \sum \tau_\alpha\nu_1^\alpha\delta\nu_2 = 0, \qquad [\tau_\alpha = \epsilon_\alpha - \tfrac{1}{2}(e_1 + e_2)], \tag{14}$$

with $\Sigma\nu_1^\alpha\delta\nu_1^\alpha = 0$ and $\Sigma\nu_2^\alpha\delta\nu_2^\alpha = 0$. Then multiplying these latter conditions by constants k_1 and k_2 and combining with (14) we deduce finally as conditions on the directions ν_1 and ν_2 for stationary $\delta\theta$ the following

$$k_1\nu_1^\alpha + \tau_\alpha\nu_2^\alpha = 0, \qquad \tau_\alpha\nu_1^\alpha + k_2\nu_2^\alpha = 0. \tag{15}$$

But multiplying the two sets of equations in (12) by ν_2^α and ν_1^α respectively and summing on the repeated index α we find $e_2 + k_3 = 0$ and $e_1 + k_3 = 0$. Hence $k_3 = -e_1 = -e_2$ and $k_3 = -(e_1 + e_2)/2$. But when we make this latter substitution into (12) these equations assume the form (15). Hence (12) is the condition for a pair of perpendicular vectors ν_1 and ν_2 to give a stationary value of $\delta\theta$ relative to *arbitrary* variations in direction.

5. Directions giving a maximum or minimum value of the angular variation. We now consider in detail the problem of finding directions ν_1 and ν_2 for which $\delta\theta$ has a maximum or minimum value. For this purpose we divide the discussion into separate cases accordingly as the ϵ's are all distinct, the values of two of the ϵ's are equal, or all ϵ's have the same value.

Case I. The ϵ's have distinct values, i.e., $\epsilon_1 > \epsilon_2 > \epsilon_3$. Suppose that the bracket expressions in (13) vanish for $\alpha = 1, 2$. Then $(\epsilon_1 + k_3)^2 = k^2$ and $(\epsilon_2 + k_3)^2 = k^2$ or $\epsilon_1 + k_3 = \pm k$ and $\epsilon_2 + k_3 = \pm k$. Both plus or both minus signs in these relations cannot be taken since we then have $\epsilon_1 = \epsilon_2$ contrary to hypothesis. Hence we must have (a) $\epsilon_1 + k_3 = k$, $\epsilon_2 + k_3 = -k$ or (b) $\epsilon_1 + k_3 = -k$, $\epsilon_2 + k_3 = k$. Taking first the equations (a) we find on solving that $k_3 = -(\epsilon_1 + \epsilon_2)/2$ and $k = (\epsilon_1 - \epsilon_2)/2$. Substituting these values of k_3 and k into the remaining bracket expression in (13), *i.e.*, the expression for which $\alpha = 3$, we have

$$(\epsilon_3 + k_3)^2 - k^2 = (\epsilon_3 + k_3 + k)(\epsilon_3 + k_3 - k) = (\epsilon_3 - \epsilon_2)(\epsilon_3 - \epsilon_1) > 0.$$

It follows from (13) that $\nu_1^3 = \nu_2^3 = 0$ so that the vectors ν_1 and ν_2 lie in the x^1, x^2 plane. From (12) with $\alpha = 1, 2$ we now find

$$k\overset{1}{\nu}_1 + (\epsilon_1 + k_3)\overset{1}{\nu}_2 = 0 \rightarrow k\overset{1}{\nu}_1 + k\overset{1}{\nu}_2 = 0, \quad \text{or} \quad \overset{1}{\nu}_1 = -\overset{1}{\nu}_2,$$

$$k\overset{2}{\nu}_1 + (\epsilon_2 + k_3)\overset{2}{\nu}_2 = 0 \rightarrow k\overset{2}{\nu}_1 - k\overset{2}{\nu}_2 = 0, \quad \text{or} \quad \overset{2}{\nu}_1 = \overset{2}{\nu}_2,$$

since $k \neq 0$. Had we used the alternative equations (b) we would have found that (12) reduces to $\nu_1^1 = \nu_2^1$, $\nu_1^2 = -\nu_2^2$ and $\nu_1^3 = \nu_2^3 = 0$. Now imposing the condition that ν_1 and ν_2 are perpendicular, namely $\nu_1^1\nu_2^1 + \nu_1^2\nu_2^2 = 0$ we obtain $-(\nu_1^1)^2 + (\nu_1^2)^2 = 0$ and hence $\nu_1^1 = \pm\nu_1^2$. Then from the relation $(\nu_1^1)^2 + (\nu_1^2)^2 = 1$ expressing the condition that ν_1 is a unit vector, we obtain $2(\nu_1^1)^2 = 1$ or $\nu_1^1 = \pm 1/\sqrt{2}$. Hence $\nu_2^1 = \pm 1/\sqrt{2}$. From the condition $(\nu_2^1)^2 + (\nu_2^2)^2 = 1$ we can now obtain $1/2 + (\nu_2^2)^2 = 1$ and hence $\nu_2^2 = \pm 1/\sqrt{2}$. Finally $\nu_1^2 = \pm 1/\sqrt{2}$. Collecting these results we have

$$\overset{1}{\nu}_1 = \pm\frac{1}{\sqrt{2}}, \quad \overset{1}{\nu}_2 = \pm\frac{1}{\sqrt{2}}, \quad \overset{2}{\nu}_1 = \pm\frac{1}{\sqrt{2}}, \quad \overset{2}{\nu}_2 = \pm\frac{1}{\sqrt{2}}$$

where any choice of algebraic signs can be made which is consistent with the requirement that the vectors ν_1 and ν_2 be perpendicular. We thus see that the possible vectors ν_1 and ν_2 can be described by saying that they bisect adjacent quadrants in the x^1, x^2 plane. If ν_1 and ν_2 bisect the first and second quadrants respectively, or if these vectors bisect the third and fourth quadrants of the x^1, x^2 plane, we find on making the appropriate substitutions of the ν's in (6) that $\delta\theta = \epsilon_1 - \epsilon_2 > 0$ in each case. When the vectors bisect the first and fourth quadrants and also when they bisect the second and third quadrants the corresponding value of $\delta\theta = -(\epsilon_1 - \epsilon_2)$.

There are two other combinations of bracket expressions in (13) which can be equated to zero and which lead to results analogous to the above. Thus stationary values of $\delta\theta$ are given by the pairs of perpendicular vectors which bisect the quadrants of the x^1, x^3 plane and the corresponding value of $\delta\theta$ is $\pm(\epsilon_1 - \epsilon_3)$. Similarly we obtain a stationary $\delta\theta$ for perpendicular vectors bisecting the quadrants of the x^2, x^3 plane, the values of $\delta\theta$ in this case being given by $\delta\theta = \pm(\epsilon_2 - \epsilon_3)$. Since there thus exist non-zero values of $\delta\theta$ it follows from the result at the beginning of this paper that the above-mentioned pairs of perpendicular vectors ν_1 and ν_2 constitute all the directions giving stationary values of $\delta\theta$.

The question now arises as to whether the above pairs of perpendicular vectors ν_1 and ν_2 give relative maximum or minimum values of $\delta\theta$. Our results show that the greatest value of $\delta\theta$ is $\epsilon_1 - \epsilon_3$ and this is determined by the pair of perpendicular vectors ν_1 and ν_2 which bisect the first and second quadrants or the third and fourth quadrants of the x^1, x^3 plane. Similarly the least value of $\delta\theta$ is $-(\epsilon_1 - \epsilon_3)$ and is determined by the vectors bisecting the first and fourth or the second and third quadrants of the x^1, x^3 plane. These are the absolute maximum and minimum values of $\delta\theta$ the existence of which has been assured by the previous discussion. There remains, however, the question whether the other stationary values of $\delta\theta$ constitute relative maxima or minima of this quantity.

This latter question is to be answered in the negative. But to show this we must calculate, to within terms of higher order, the change $\Delta\delta\theta$ in $\delta\theta$ determined

by the variations $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$. Take the case for which the vectors ν_1 and ν_2 bisect the first and second quadrants of the x^1, x^2 plane and for which $\delta\theta=\epsilon_1-\epsilon_2$ as shown above. The components of these vectors are then given by $\nu_1^1=\nu_1^2=1/\sqrt{2}$, $\nu_1^3=0$ and $-\nu_2^1=\nu_2^2=1/\sqrt{2}$, $\nu_2^3=0$. Values of ν_1^α and ν_2^α in the neighborhood of these values can be represented by the spherical coordinates of points on the unit sphere with center at the origin. Thus

$$\overset{1}{\nu}_1 = \sin\phi_1\cos\phi_2, \qquad \overset{2}{\nu}_1 = \sin\phi_1\sin\phi_2, \qquad \overset{3}{\nu}_1 = \cos\phi_1,$$
$$\overset{1}{\nu}_2 = \sin\phi_3\cos\phi_4, \qquad \overset{2}{\nu}_2 = \sin\phi_3\sin\phi_4, \qquad \overset{3}{\nu}_2 = \cos\phi_3.$$

To find the variations $\delta\nu_1^\alpha$ and $\delta\nu_2^\alpha$ in terms of the arbitrary variations of the ϕ's we expand the trigonometric functions in the above equations about the following initial values: $\phi_1=\pi_2$, $\phi_2=\pi/4$, $\phi_3=\pi/2$, and $\phi_4=\pi/2+\pi/4$. We thus find

$$\sin\phi_1 = 1-\frac{1}{2}(\delta\phi_1)^2+\cdots$$
$$\cos\phi_2 = \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\delta\phi_2-\frac{1}{2\sqrt{2}}(\delta\phi_2)^2+\cdots$$
$$\sin\phi_2 = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\delta\phi_2-\frac{1}{2\sqrt{2}}(\delta\phi_2)^2+\cdots$$
$$\cos\phi_1 = -\delta\phi_1+\cdots$$
$$\sin\phi_3 = 1-\frac{1}{2}(\delta\phi_3)^2+\cdots$$
$$\cos\phi_4 = -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\delta\phi_4+\frac{1}{2\sqrt{2}}(\delta\phi_4)^2+\cdots$$
$$\sin\phi_4 = \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\delta\phi_4-\frac{1}{2\sqrt{2}}(\delta\phi_4)^2+\cdots$$
$$\cos\phi_3 = -\delta\phi_3+\cdots.$$

Now $\cos\theta=\sum\nu_1^\alpha\nu_2^\alpha$. The right member of this equation can be expressed in terms of the $\delta\phi$'s by the above substitutions. In the left member put $\theta=\pi/2+\Delta\theta$ and expand about $\pi/2$. This leads to the relations

$$\cos\theta = -\Delta\theta = \delta\phi_2-\delta\phi_4+\delta\phi_1\delta\phi_3,$$

to within the order of terms which we need to consider. Turning now to the equation (1) we replace $\delta\theta$ by $\delta\theta+\Delta\delta\theta$ and express $\sin\theta$ in terms of $\Delta\theta$ by expansion about $\pi/2$. To within terms of the second order in the variations $\delta\phi$ the left member of (1) then becomes

$$\Delta\delta\theta+(\epsilon_1-\epsilon_2)-\tfrac{1}{2}(\epsilon_1-\epsilon_2)\left[(\delta\phi_2)^2-2\delta\phi_2\delta\phi_4+(\delta\phi_4)^2\right]$$

when use is made of the above value of $\Delta\theta$ and the fact that $\delta\theta=\epsilon_1-\epsilon_2$. Finally

making use of the above value of $\cos\theta$ in terms of the $\delta\phi$'s and also making the above substitutions for the ν_1^α and ν_2^α in the right member of (1), we find, after reduction, that

$$-2\Delta\delta\theta = A(\delta\phi_1)^2 - 2B\delta\phi_1\delta\phi_3 + 2A(\delta\phi_2)^2 + A(\delta\phi_3)^2 + 2A(\delta\phi_4)^2$$

to within quadratic terms in the variations $\delta\phi$. Here $A=\epsilon_1-\epsilon_2>0$ and $B=(\epsilon_1-\epsilon_3)+(\epsilon_2-\epsilon_3)>0$. The matrix of the coefficients of this (symmetric) quadratic form in the $\delta\phi$'s is

$$\begin{Vmatrix} A & 0 & -B & 0 \\ 0 & 2A & 0 & 0 \\ -B & 0 & A & 0 \\ 0 & 0 & 0 & 2A \end{Vmatrix}$$

and the algebraic signs of the successive diagonal determinants are $+, +, -, -$. Hence in accordance with the usual test this form is neither positive nor negative definite, and hence the value $\delta\theta=\epsilon_1-\epsilon_3$ under consideration is neither a relative maximum nor minimum. It follows that a similar conclusion holds for the negative of this value of $\delta\theta$ namely $\delta\theta=-(\epsilon_1-\epsilon_3)$.

By rotating the coordinate axes as indicated by $(x^3, x^1, x^2)\rightarrow(x^1, x^2, x^3)$ and making the corresponding substitution on the quantities ϵ_α namely $(\epsilon_3, \epsilon_1, \epsilon_2)\rightarrow(\epsilon_1, \epsilon_2, \epsilon_3)$ we can pass from the case just treated to that for which the vectors ν_1 and ν_2 bisect the quadrants of the x^2, x^3 plane. Since the matrix derivable from the above matrix by this substitution on the ϵ's is neither positive nor negative definite, it follows that the values $\delta\theta=\pm(\epsilon_2-\epsilon_3)$ corresponding to vectors ν_1 and ν_2 bisecting the quadrants of the x^2, x^3 plane are neither relative maxima nor minima. However, the above matrix passes into one corresponding to a definite quadratic form under the substitution $(\epsilon_3, \epsilon_1, \epsilon_3)\rightarrow(\epsilon_2, \epsilon_3, \epsilon_1)$ which gives an explicit proof of the fact already observed that the values $\delta\theta=\epsilon_1-\epsilon_3$ and $\delta\theta=-(\epsilon_1-\epsilon_3)$ are respectively maximum and minimum values of $\delta\theta$.

We can now sum up our results for Case I as follows. *Vectors ν_1 and ν_2 bisecting the first and second or the third and fourth quadrants of the x^1, x^3 plane give a maximum $\delta\theta=(\epsilon_1-\epsilon_3)$ while those bisecting the first and fourth or the second and third quadrants of this plane give a minimum value $\delta\theta=-(\epsilon_1-\epsilon_3)$. Vectors ν_1 and ν_2 bisecting adjacent quadrants of the x^1, x^2 or x^2, x^3 planes give stationary values of $\delta\theta$ but these values are neither relative maxima nor minima.* In making this statement it is of course to be understood (as also in the following) that the vectors ν_1 and ν_2 have their initial points at the origin of the rectangular coordinate system for which $e_{\alpha\beta}=\epsilon_\alpha\delta_{\alpha\beta}$ and that moreover the coordinate axes have been enumerated so that the inequalities $\epsilon_1>\epsilon_2>\epsilon_3$ hold.

Case II. *Two of the ϵ's are equal.* This case breaks up into the following subcases, namely (α) $\epsilon_1=\epsilon_2>\epsilon_3$ and (β) $\epsilon_1>\epsilon_2=\epsilon_3$. We shall give the discussion in detail under the assumption (α).

Since two of the bracket expressions in (13) must be equal to zero we must

have $(\epsilon_1+k_3)^2-k^2=0$ and hence (a) $\epsilon_1=k-k_3$ or (b) $\epsilon_1=-k-k_3$. If $k=0$ there is no distinction between (a) and (b). We treat this special case first. Then equations (12) reduce to $(\epsilon_3+k_3)\nu_2^3=0$ and $(\epsilon_3+k_3)\nu_1^3=0$. But $\epsilon_3+k_3=\epsilon_3-\epsilon_1<0$ and hence $\nu_1^3=\nu_2^3=0$. Hence ν_1 and ν_2 can be taken as any pair of perpendicular vectors in the x^1, x^2 plane.

Now assume $k\neq 0$ and suppose (a) to hold. Then equations (12) become

$$(16)\qquad \begin{cases} \nu_1^1+\nu_2^1=0. & \nu_1^2+\nu_2^2=0, \\ k\nu_1^3+(\epsilon_3+k_3)\nu_2^3=0, & (\epsilon_3+k_3)\nu_1^3+k\nu_2^3=0. \end{cases}$$

If we combine the equation expressing the fact that the vectors ν_1 and ν_2 are perpendicular with the equations (16) we find

$$k(\nu_1^1)^2+k(\nu_1^2)^2+(\epsilon_3+k_3)(\nu_1^3)^2=0.$$

But,

$$k(\nu_1^1)^2+k(\nu_1^2)^2+k(\nu_1^3)^2=k,$$

since ν_1 is a unit vector, and hence

$$[k-(\epsilon_3+k_3)](\nu_1^3)^2=k \quad \text{or} \quad (\epsilon_1-\epsilon_3)(\nu_1^3)^2=k.$$

Since $\epsilon_1-\epsilon_3\neq 0$ and $k\neq 0$ it follows that $\nu_1^3\neq 0$. Hence the values of the quantities ν_1^3 and ν_2^3 satisfying the two equations in the second line of (16) cannot both be zero and hence the determinant of the coefficients in these equations must vanish. But this determinant is

$$(\epsilon_3+k_3)^2-k^2=(\epsilon_3+k_3-k)(\epsilon_3+k_3+k)=(\epsilon_3-\epsilon_1)(\epsilon_3+k_3+k)=0.$$

Then since $\epsilon_3-\epsilon_1\neq 0$ we must have $\epsilon_3+k_3+k=0$. Hence the two equations in the second line of (16) reduce to the single equation $\nu_1^3=\nu_2^3$.

We have thus shown under the above condition (a) that equations (12) reduce to $\nu_1^1=-\nu_2^1$, $\nu_1^2=-\nu_2^2$ and $\nu_1^3=\nu_2^3$. Under condition (b) we find in a similar manner that we may also have $\nu_1^1=\nu_2^1$, $\nu_1^2=\nu_2^2$ and $\nu_1^3=-\nu_2^3$. These results can be expressed as follows: The vectors ν_1 and ν_2 can be taken as any pair of perpendicular vectors which issue from the origin of the coordinate system and which lie along generators of the complete circular cone C, the generators of which make an angle of $\pi/4$ with the axis, and whose axis coincides with the x^3 coordinate axis. Under condition (a) both vectors ν_1 and ν_2 lie in the same nappe of the cone, while under condition (b) the vectors ν_1 and ν_2 lie in different nappes.

Calculation of the values of $\delta\theta$ corresponding to the above determinations of the vectors ν_1 and ν_2 is readily carried out and leads to the following results. Corresponding to $k=0$ for which ν_1 and ν_2 are perpendicular vectors in the x^1, x^2 plane we find $\delta\theta=0$. When ν_1 and ν_2 have the determination under condition (a) above, we have $\delta\theta=\epsilon_1-\epsilon_3$ and when the vectors ν_1 and ν_2 are determined according to condition (b) then $\delta\theta=-(\epsilon_1-\epsilon_3)$. The above pairs of perpendicular vectors ν_1 and ν_2 constitute all pairs of directions for which $\delta\theta$ has stationary values. Also we have max. $\delta\theta=\epsilon_1-\epsilon_3$ and min. $\delta\theta=-(\epsilon_1-\epsilon_3)$.

We can readily observe that the value $\delta\theta = 0$ determined by any pair of perpendicular vectors $\nu_1 = \nu_1^0$ and $\nu_2 = \nu_2^0$ is neither a relative maximum nor minimum. To show this let us first determine all pairs of perpendicular vectors ν_1 and ν_2 for which $\delta\theta = 0$. The required conditions on such vectors are obtained by equating the right members of (6) to zero. This gives

$$\sum \epsilon_\alpha \overset{\alpha}{\nu_1}\overset{\alpha}{\nu_2} = \epsilon_1(\overset{1}{\nu_1}\overset{1}{\nu_2} + \overset{2}{\nu_1}\overset{2}{\nu_2}) + \epsilon_3 \overset{3}{\nu_1}\overset{3}{\nu_2} = (\epsilon_3 - \epsilon_1)\overset{3}{\nu_1}\overset{3}{\nu_2} = 0.$$

Since $\epsilon_3 - \epsilon_1 \neq 0$ it follows that $\nu_1^3 = 0$ or $\nu_2^3 = 0$, *i.e.* one of the vectors ν_1 or ν_2 must lie in the x^1, x^2 plane. Now take a pair of perpendicular vectors ν_1 and ν_2 lying along generators in the same nappe of the cone C, for example in the nappe lying in the semispace $x^3 \geqq 0$. For these vectors we then have $\delta\theta = \epsilon_1 - \epsilon_3 > 0$. Displace ν_1 and ν_2 continuously, but without altering the condition of perpendicularity into ν_1^0 and ν_2^0 respectively in such a way that the end points of the vectors always lie in the same semispace with respect to the x^1, x^2 plane. It follows from the result just proved that we shall then have $\delta\theta > 0$ throughout this displacement. Similarly if we start with perpendicular vectors ν_1 and ν_2 lying along generators of C, but in different nappes of C, and displace them continuously under the condition of perpendicularity into ν_1^0 and ν_2^0 in such a way that the end point of each vector always remains in the same semispace with respect to the x^1, x^2 plane, then we shall have $\delta\theta < 0$ throughout the displacement. Hence in any neighborhood of ν_1^0 and ν_2^0 there exist vectors ν_1 and ν_2 for which $\delta\theta > 0$ and vectors ν_1 and ν_2 for which $\delta\theta < 0$. It follows that the value $\delta\theta = 0$ determined by ν_1^0 and ν_2^0 is neither a relative maximum nor minimum.

Corresponding results are obtained under the assumption (β) for which $\epsilon_1 > \epsilon_2 = \epsilon_3$. Now instead of the above cone C we have the analogous cone C' with axis along the x^1 coordinate axis. We give the results in the following statement which contains the complete results for Case II. *If $\epsilon_1 = \epsilon_2 > \epsilon_3$ perpendicular vectors ν_1 and ν_2 lying along generators in the same nappe of the cone C yield max. $\delta\theta = \epsilon_1 - \epsilon_3$ while those perpendicular vectors along generators in different nappes of C give min. $\delta\theta = -(\epsilon_1 - \epsilon_3)$. A stationary value $\delta\theta = 0$ is given by any pair of perpendicular vectors ν_1 and ν_2 in the x^1, x^2 plane, this value being neither a relative maximum nor minimum. If $\epsilon_1 > \epsilon_2 = \epsilon_3$ those perpendicular vectors ν_1 and ν_2 which lie along generators in different nappes of the cone C' give the max. $\delta\theta = \epsilon_1 - \epsilon_3$ and perpendicular vectors ν_1 and ν_2 along generators in the same nappe of C' give the min. $\delta\theta = -(\epsilon_1 - \epsilon_2)$. When the vectors ν_1 and ν_2 are any pair of perpendicular vectors in the x^2, x^3 plane we obtain a stationary value of $\delta\theta = 0$ which is neither a relative maximum nor a minimum.*

Case III. *All* ϵ's *have the same value.* This case can be handled quickly. Let ν_1 and ν_2 be any two unit vectors not necessarily perpendicular. Then since $\epsilon_1 = \epsilon_2 = \epsilon_3$ it is immediately seen that the right member of (1) is equal to zero. *Hence $\delta\theta = 0$ for arbitrary directions ν_1 and ν_2.*

This completes the determination of all directions giving angles of maximum and minimum variation under small displacements.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A MATTER OF MOTIVATION

H. V. CRAIG, University of Texas

It is well known that vector analysis is admirably suited to the exposition of a miscellany of topics in elementary mathematics and advanced calculus. Usually, the vectorial treatments are short, simple, and *well motivated*. A case in point that so far as the writer knows has escaped attention is the Lagrange multiplier method for determining critical points. Roughly, the problem involved, in its simplest form, is to find a necessary condition that a point P on a surface $S(x, y, z)=0$ give a function $F(x, y, z)$ an extreme value relative to the surface—we assume that all functions are of class c' and that $\nabla S \neq 0$. Evidently, if P is such a point; and C $(x=x(s), y=y(s), z=z(s))$ is a curve passing through P and lying entirely on $S=0$; then the derivative dF/ds vanishes at P. The traditional development of Lagrange's method of attack is fairly concise but emphatically it is not well motivated. The fundamental expression $F(x, y, z)+\lambda S(x, y, z)$ is essentially extracted from thin air. With regard to its introduction Osgood* says: "The motif lies in the purely algebraic situation which arises when we do this thing."

Let us now examine the problem from the standpoint of vector analysis. If s is arc length on C and τ is the unit tangent vector to C at P, then since F has an extreme value at P and since $S(x, y, z)=0$ on the surface, we have

$$0=\frac{dF}{ds}\Big| \text{ at } P=\nabla F\cdot\tau\Big| \text{ at } P; \qquad 0=\frac{dS}{ds}\Big| \text{ at } P=\nabla S\cdot\tau\Big| \text{ at } P.$$

But C is any curve on the surface and therefore τ is any unit vector tangent to the surface at P. Consequently, either ∇F vanishes at P or it, as well as ∇S, is normal to the surface. In either case we have at P, $\nabla F=\lambda\nabla S$. This last equation together with $S=0$, comprises the multiplier method.

The case of two added conditions $S_1(x, y, z)=S_2(x, y, z)=0$ may be treated similarly. We suppose that these equations determine a curve. The choice of C is now limited to this curve, and we conclude that if ∇F at P is not zero, then ∇F, ∇S_1, and ∇S_2 are each perpendicular to τ. In either case we have at P, $\nabla F=\lambda_1\nabla S_1+\lambda_2\nabla S_2$, assuming ∇S_1 and ∇S_2 are not zero at P.

* See W. F. Osgood, Advanced Calculus, Macmillan, 1925, p. 180.

ON THE CALCULATION OF BOND YIELDS

H. D. LARSEN, University of New Mexico

An important problem in the mathematics of finance is that of determining the yield rate on a bond given its purchase price. Tables of bond values are published for yield rates proceeding by one twentieth of 1%, and simple interpolation in these tables furnishes an approximation to the yield which suffices for most purposes. When extensive tables of bond values are not at hand, recourse is made to annuity tables such as those generally found in textbooks on the mathematics of finance. However, simple interpolation in the latter tables does not always give the accuracy desired. Many methods have been proposed to remedy this situation. In general, these methods depend on formulas of finite differences or on successive approximations, and involve considerable computing. It is the purpose of this note to present a simple recursion formula for calculating successive approximations to the yield of a bond. The accuracy of the formula is limited only by the accuracy of the logarithm tables used in its connection, a five-place table supplemented by a seven-place table of log $(1+i)$ being sufficient to give the yield rate to at least six places of decimals.

Consider a bond redeemable at the end of n periods (not necessarily at par), and carrying a dividend at the rate of g per period per unit of its redemption price. Let the bond be purchased at a premium k per unit on its redemption price, where $|k| < 1$. The restriction on k is not serious, as it requires only that the purchase price of the bond be less than twice the redemption price, the usual situation in practice. If i denotes the required yield rate per period, then

$$k = (g - i)a, \tag{1}$$

where

$$a = a_{\overline{n}|} \quad \text{at} \quad i.$$

Solving (1) for i, we obtain

$$i = g - ka^{-1}. \tag{2}$$

Equation (2) suggests the recursion formula,

$$i_{m+1} = g - ka_m^{-1}, \tag{3}$$

where $a_m = a_{\overline{n}|}$ at i_m, and i_1 is any convenient first approximation to i. In order to prove that the successive approximations given by (3) actually converge to i, it suffices to show that

$$\lim_{m\to\infty} |\epsilon_{m+1}| = \lim_{m\to\infty} |i_{m+1} - i| = 0.$$

From (2) and (3),

$$\epsilon_{m+1} = k(a^{-1} - a_m^{-1}). \tag{4}$$

If $i > i_m$, then

$$0 < a^{-1} - a_m^{-1} = (i + s_{\overline{n}|}^{-1} \text{ at } i) - (i_m + s_{\overline{n}|}^{-1} \text{ at } i_m) < i - i_m,$$

since

$$s_{\overline{n}|}^{-1} \text{ at } i < s_{\overline{n}|}^{-1} \text{ at } i_m; \qquad i > i_m.$$

Similarly, if $i < i_m$,

$$0 < a_m^{-1} - a^{-1} < i_m - i.$$

In either case,

$$|a^{-1} - a_m^{-1}| < |i_m - i| = |\epsilon_m|,$$

whence, by (4),

$$|\epsilon_{m+1}| < |k| \cdot |\epsilon_m|. \tag{5}$$

A successive application of (5) gives

$$|\epsilon_{m+1}| < |k|^m \cdot |\epsilon_1|,$$

so that, since $|k| < 1$,

$$\lim_{m \to \infty} |\epsilon_{m+1}| = 0.$$

The above discussion discloses the nature of the errors, ϵ_m. It is clear from (5) that *every* approximation is better than the preceding one, and that the rapidity of the convergence increases as $|k|$ decreases. Further, since $a_m^{-1} - a^{-1}$ has the same sign as $i_m - i = \epsilon_m$, it follows from (4) that ϵ_{m+1} and ϵ_m have like signs or opposite signs according as $k < 0$ or $k > 0$. Thus, if $k < 0$, the successive approximations are all too large or all too small; if $k > 0$, the succcessive approximations are alternately too large and too small. It should be noted also that a mistake in any particular approximation does not vitiate the ultimate convergence of the subsequent approximations. The incorrect value merely becomes a new first approximation to i.

A first approximation to i may be obtained in a variety of ways. In the example below, i_1 is computed from the formula,*

$$i_1 = \frac{g - k/n}{1 + (n+1)k/2n}, \tag{6}$$

which is derived from (2) by expanding a^{-1} in a power series and neglecting all terms above the first degree in i. It is easy to show that the error ϵ_1 resulting from the use of (6) always has the same sign as k. Consequently, if $k < 0$, then $\epsilon_1 < 0$, $\epsilon_2 < 0$, $\epsilon_3 < 0$, *etc.*; if $k > 0$, then $\epsilon_1 > 0$, $\epsilon_2 < 0$, $\epsilon_3 > 0$, *etc.*

As an example, let it be required to find the yield on a bond bearing interest

* Ralph Todhunter, Institute of Actuaries Text-Book, Part I. The Theory of Compound Interest and Annuities Certain, New Edition, revised. Charles and Edwin Layton, London, 1915, pp. 104–105.

at 4½ per cent, payable half-yearly, redeemable in 25 years at 112½, and bought at a price of 120. (Todhunter, p. 105.) Here,

$$n = 50; \qquad g = 0.0225/112.5 = 0.02; \qquad k = 7.5/112.5 = 1/15.$$

Hence,

$$i_1 = \frac{0.02 - 1/750}{1 + (51/100)(1/15)} = 0.018053,$$

$$a_1^{-1} = \frac{0.018053}{1 - (1.018053)^{-50}} = 0.030535,$$

$$i_2 = 0.02 - (1/15)(0.030535) = 0.017964,$$

$$a_2^{-1} = \frac{0.017964}{1 - (1.017964)^{-50}} = 0.030477,$$

$$i_3 = 0.02 - (1/15)(0.030477) = 0.017968.$$

The value of i_3 agrees with the true yield rate to six places of decimals.

AN IMPROPER INTEGRAL

E. J. Camp, Macalester College

In Osgood's *Advanced Calculus** the integral

$$\int_0^\infty \frac{\cos mx}{1 + x^2}\,dx$$

appears on page 490 along with a suggested transformation for its evaluation. If one performs the transformation and then makes use of two other improper integrals which have been worked out on previous pages, the integral can be evaluated. The transformation is rather artificial and is not one that is suggested by a sequence of logical steps. The method of this paper will be to evaluate the above integral by the use of a second order differential equation.

Let the function $u(m, \beta)$ be defined by the equation

$$u(m, \beta) = \int_0^\infty \frac{e^{-\beta x}\cos mx}{x^2 + 1}\,dx, \tag{1}$$

where β is regarded as constant. The derivatives† of this function are

$$\frac{du}{dm} = \int_0^\infty \frac{-xe^{-\beta x}\sin mx}{x^2 + 1}\,dx, \tag{2}$$

$$\frac{d^2u}{dm^2} = \int_0^\infty \frac{-x^2e^{-\beta x}\cos mx}{1 + x^2}\,dx. \tag{3}$$

If equation (1) is subtracted from equation (3) the result is the equation

* See W. F. Osgood, Advanced Calculus, Macmillan, 1925, p. 490.

† The integrals 1, 2, and 3 are uniformly convergent and so the differentiation under the integral sign is justified.

$$\frac{d^2u}{dm^2} - u = \frac{-\beta}{\beta^2 + m^2}. \tag{4}$$

Solving equation (4) by the method of variation of parameters, we get

$$u = c_1e^m + c_2e^{-m} + \tfrac{1}{2}e^m\int_0^m \frac{-\beta e^{-t}}{\beta^2 + t^2}dt + \tfrac{1}{2}e^{-m}\int_0^m \frac{\beta e^t}{\beta^2 + t^2}dt, \quad m > 0.* \tag{5}$$

If m is set equal to 0 in equation (5) the right member has the value c_1+c_2. The same function in the form (1) has the value

$$\int_0^\infty \frac{e^{-\beta x}}{x^2 + 1}dx.$$

The right member of equation (2) has the value 0 when $m=0$. The derivative du/dm obtained from equation (5) is c_1-c_2 for $m=0$. These considerations lead to the equations

$$\begin{aligned} c_1 + c_2 &= \int_0^\infty \frac{e^{-\beta x}}{x^2 + 1}dx, \\ c_1 - c_2 &= 0, \end{aligned} \tag{6}$$

the solutions of which are

$$c_1 = c_2 = \frac{1}{2}\int_0^\infty \frac{e^{-\beta x}}{x^2 + 1}dx. \tag{7}$$

The problem is now solved except for evaluating the integrals in (7) and (5) for $\beta=0$. The integrals in (7) have the value $\pi/4$.

The integrals in equation (5) which belong to the particular integral will now be evaluated. Thus if we set $t=\beta\tau$ we have

$$\frac{1}{2}e^m\int_0^m \frac{-\beta e^{-t}}{\beta^2 + t^2}dt = -\frac{1}{2}e^m\int_0^{m/\beta} \frac{e^{-\beta\tau}}{1 + \tau^2}d\tau$$

and the limit of this integral for $\beta=0$ is $-\frac{1}{2}e^m(\pi/2)$. Similarly

$$\lim_{\beta=0}\left[\frac{1}{2}e^{-m}\int_0^m \frac{\beta e^t}{\beta^2 + t^2}dt\right] = \frac{\pi}{4}e^{-m}.$$

On substituting each of these values in equation (5) we have

$$\lim_{\beta=0} u(m,\beta) = \frac{\pi}{4}e^m + \frac{\pi}{4}e^{-m} - \frac{\pi}{4}e^m + \frac{\pi}{4}e^{-m} = \frac{\pi}{2}e^{-m}, \quad m > 0.$$

It follows from the third footnote that where m is any real number,

$$\int_0^\infty \frac{\cos mx}{1 + x^2}dx = \frac{\pi}{2}e^{-|m|}.$$

* If m is negative then the equation will be of the same form with m replaced by $|m|$.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Kappa Mu Epsilon, Kansas State Teachers College, Pittsburg

The activities for the year consisted of six open meetings held at the homes of faculty members, two picnics, and two closed fraternity meetings. Twelve new members were initiated at the summer fraternity meeting and forty-six new members were initiated at the mid-year meeting. The programs of these meetings were followed by the initiation services. Attendance was greatly increased by the presence of a Navy V-12 unit on the campus.

Included among the topics discussed at meetings were:

The practical application of mathematics to engineering, by E. R. Sutterfield, draftsman at the Pittsburg-McNally Manufacturing Corporation

The rehabilitation program at K. S. T. C., by Professor W. H. Matthews, Director of Adult Education

Types of non-euclidean geometry, by Professor R. G. Smith

Recent discoveries in astronomy, by Professor J. A. G. Shirk

Mathematics in the war, by Marie Cowley, Instructor of Physical Science

Mathematics after the war, by Professor F. C. German.

Student members of the club talked on the following topics:

Mathematical prodigies

Trisection of angles

The history of mathematics from 600 B.C. to 200 B.C.

The history of mathematics to the present day.

Officers were: President Archimedes, Helen Kriegsman; Vice-President Plato, Mrs. Mildred Bradshaw; Secretary Lagrange, Mary Lou Ralston; Treasurer Thales, Virginia Prussing; Corresponding Secretary, Professor W. H. Hill, K. S. T. C., Pittsburg, Kansas; and Sponsor, Professor J. A. G. Shirk.

Mathematics Club, Brown University

Monthly meetings were held at eight o'clock in the evening, and were followed by refreshments. Papers presented during the fall semester were as follows:

Ancient number systems, with lantern slides, by Professor Otto Neugebauer, R. P. Breeding presiding

The isograph—a mechanical root finder, with film and lantern slides, by R. E. Jacobson, Jr., S. L. Ehrlich, presiding

Skeleton division, by Anna Irene La Banca, and

Practical applications of special mathematical curves, by Samuel Millman, Winifred O'Connell presiding

From March to May, 1944, the program included the following talks:

On some properties of a circle, by Professor J. D. Tamarkin, R. P. Breeding, presiding

The cell of the honey bee, by Shirley Gallup, and

The number pi, by M. E. Carlson; Frances Weeden presiding

Arithmetical prodigies, by L. M. Pease, and

The duodecimal system of numeration, by C. V. Harding, Jr., Alice Nancy Noyes presiding

During the summer semester 1944, three papers were presented:

Growth and mathematics, by Professor J. Walter Wilson of the Department of Biology, Shirley Gallup presiding

Mathematical fallacies, by F. W. Suffa, and

Map coloring problem, by A. B. Novikoff, C. V. Harding presiding.

A club picture was taken after the April meeting, and picnics were scheduled for May 27 and September 9. The club was directed by a Committee on Program and Arrangements, consisting of a Faculty Representative: Professor R. C. Archibald; a Student Chairman: R. P. Breeding (until spring '44), Shirley Marily Gallup (summer '44); and the following committee members: Barbara Cotter, Stanley Ehrlich, Shirley Gallup, Frances Weeden (until spring '44); and Leslie Carson, Stanley Ehrlich, Clifford Harding, Jr., Evelyn Lindsay (summer '44).

Kappa Mu Epsilon, Texas Technological College, Lubbock

A membership of fifty-eight active members, including twenty-nine new initiates, held regular meetings on the second Thursday of each month. At the first meeting, the local officers discussed the mathematicians whom they represented:

Lobatchevsky, by Virginia Bowman, president

Agnesi, by Brac Biggers, vice-president

Noether, Cayley, by Betty Grace Pugh, secretary-treasurer

Descartes, by Lida B. May, corresponding secretary.

Initiations, followed by refreshments were held in November. The December meeting at the home of Dr. and Mrs. F. W. Sparks, was a Christmas party, including games, puzzles, card tricks, refreshments, and Christmas carols. A very successful talk was presented at the February meeting, entitled

The sun, our source of light and energy, by Donald Robbins.

The spring initiation banquet was held at Cheri Casa. Projects for the year included the sending of Christmas cards to former members now in the armed service, and to past presidents of the chapter; and the purchase of War Savings Stamps and a twenty-five dollar War Bond. Plans have been initiated for making an annual award to the outstanding freshman in mathematics. Officers elected for 1944–45 are: President, Betty Grace Pugh; Vice-President, Tom Hassell; Secretary, Beverly Price; Treasurer, Betty Jane Morris; Corresponding Secretary, Lida B. May; Sponsor, Dr. D. L. Webb.

Mathematics Club, University of Dayton

Weekly meetings were held during the summer 1943, and semimonthly meetings during the next three terms, at which members presented papers. At the last meeting of each term the *Dean of Science Award* was presented to the student who delivered the most interesting paper before the Club. The awards were copies of the following books: *Men of Mathematics* by E. T. Bell (summer and fall 1943); *What is Mathematics?* by Courant and Robbins (winter 1944); *Isaac Newton—a Biography*, by L. T. More (summer 1944). The *Mathematics Club Alumni Awards for Excellence in Advanced Mathematics* were conferred at the December commencement upon Richard O'Brien (a junior), and at the summer commencement upon Lloyd Weeks (a senior) and Dennis Griffin (a junior).

Student papers during the summer term 1943, were:

The life of Leonard Euler, by William Fitzgibbons

The life of Sir Isaac Newton, by William McHugh

The nine point circle theorem, by Godfrey Kampner

Duodecimal arithmetic, by Allan Braun, prize winner

The history of the Mathematics Club, by Patrick Murphy, given as the vice-president's charge to the ten new members at the formal ceremony.

At the final meeting an alumnus of the club, Mr. Jack Homan, spoke on *The theory of mechanical vibration.*

Student papers during the fall term 1943 were:

Non-euclidean geometries, by Richard Welsh, prize winner

Mathematics and biology, by Edward Buescher

The origin and history of pi, by Robert Reef

Three classical problems of Greek antiquity, by Margaret Magin

The nature and importance of zero, by Dennis Griffin.

Two faculty talks were presented during the fall as follows:

The nature and history of the mathematical tripos, by Dr. K. C. Schraut

The aims of education and research in mathematics, by Professor J. L. Synge of Ohio State University.

Student papers during the winter term 1944 were:

The algebra of sets, by Dennis Griffin

The theory of groups, by Edward Buescher

Curves, by George Igel

The graphical solution of algebraic problems, by Edward Freeh

Mathematical recreations, by David Tom

The Mathematics Club, past, present, and future, by James Schuler, prize winner, given as the vice-president's charge to the eleven new initiates, and subsequently published in the University's literary magazine, the *Exponent.*

Two faculty talks were presented during the spring as follows:

Caustic curves, by Professor K. C. Schraut

Geometrography, by Professor W. J. Bellmer.

Student papers during the summer term 1944 were:

The application of vectors to the proof of plane geometry theorems, by David Timmer

The real number system, by Dennis Griffin, prize winner

The theory of matrices, by Lloyd Weeks

The history of numerals, by Allan Braun

The future of the mathematics club, by George Igel, given as the charge to eleven new members at the formal initiation ceremony.

Two faculty papers were:

Gombert's curve, by Mr. C. G. Peckham

The theory of numbers, by Professor I. A. Barnett of the University of Cincinnati.

Social events of the year were a camp supper at Hills and Dales Park during the fall term, a banquet at Wishing Well Inn during the winter term, and a summer picnic at Madden Park.

Officers of the club during the several terms were the following: Presidents: William Fitzgibbons (S'43), William McHugh (F'43), David Tom (W'44), George Igel (S'44); Vice-Presidents: Patrick Murphy (S, F'43), Richard Welsh (F'43), James Schuler (W'44), Allen Braun (S'44); Secretary-Treasurers: Edward Buescher (S, F'43), Allan Braun (W'44, Sec'y S'44), Kenneth Trimbach (Treas. S'44). Professor K. C. Schraut was Faculty Adviser.

Pi Mu Epsilon, Northwestern University

At the *Mathematics Festival*, a whole day devoted to mathematics, the Illinois Beta Chapter of *Pi Mu Epsilon* was installed at Northwestern University on May 20, 1944. The *Mathematics Club* which had been active at Northwestern University for many years served as the nucleus for the honorary group. Several hundred people helped celebrate this installation at the *Mathematics Festival* and heard distinguished guest speakers. The banquet culminated the day's activities and included the initiation of 70 charter members, 15 of whom are faculty members, as well as the presentation of the charter by Professor Tomlinson Fort of Lehigh University, Director-General of *Pi Mu Epsilon*. Papers presented during the day's program were:

Adventure vs. mathematics, by Professor E. R. Smith, Iowa State College

The concept of area in plane geometry, by Professor C. C. MacDuffee, University of Wisconsin

Alignment charts and projective geometry, by Professor L. R. Ford, Illinois Institute of Technology

Stirling's numbers and Taylor's formula, by Professor Tomlinson Fort, Lehigh University

Demonstration and heuristic reasoning in mathematics, by Professor L. M. Graves, University of Chicago.

Among the papers presented to the Club during 1943 were

The mathematical theory of utility, by Professor H. T. Davis

The analytic theory of continued fractions, by Professor H. S. Wall

Geographic maps and their scales, by Professor D. R. Curtiss
Systems of linear differential equations, by Dean Walter Bartky
Some interesting facts about integers, by Professor L. W. Griffiths.

Papers presented to the Chapter at meetings in the summer of 1944, scheduled to fit in with the accelerated programs of both the civilian students and Navy V-12 alike, were:

Pi by chance, by Dr. Elliot Buell
The dynamics of the solar system, by Professor Oliver Lee
Alexandria, home of mathematics, by Professor H. T. Davis.

Officers for 1943 were: President, Frank Morehouse; Vice-President, James Murrin; Secretary-Treasurer, Beth Henry; Social Chairman, Alice Christiansen; Assistant Social Chairman, Marilyn Arms; Faculty Adviser, H. A. Simmons. Officers for 1944–45 are: President, Isabel Emma Barrett; Vice-President, John Pederson; Secretary, Elizabeth McDonald; Treasurer, Paul Axt; Social Chairman, Priscilla Mae Williams. Correspondence should be addressed to Professor E. H. C. Hildebrandt.

Mathematics Club, University of Buffalo

Three double-feature meetings were held in each semester, at which the following programs were presented:

The nine point circle, by Marjorie Easterbrook, and
The theory of relativity, by Anatole Shapiro.
The value and uses of the slide rule, by Judy Ullmann, and
Some mathematical puzzles, presented for the students to work, by Lois Obenauer.

Locks, by Jessie Brown and Joanne Yunker, featured by a display of present day locks, and slides of old time locks; and

A Chinese mystery, presented for the audience to solve, by John Euller.

At the February meeting guest speakers were the following former students who described how their background in mathematics had helped them in their work: Ruth Euller Heintz, a Spencer Lens worker; Virginia McCausland, a Curtiss Cadet; and Joan Searles, a teacher.

The cloudchamber, a paper by John Euller, was followed at the March meeting by some tricky mathematics problems, presented by Phyllis Valentine and Jane Noller.

How maps are made, was presented jointly by Audrey Strabel and Richard Salemi at the final April meeting, to which students of various high schools were invited. A discussion followed the lecture.

Social events sponsored by the Club during the year were a bowling party (at which scores ranged from 50 to 150); and a scavenger hunt held at the home of Dr. Gehman, at which the prize-winning team consisted of John Euller, Jane Noller, Marie Alessi, and Mary Jane Gill.

Officers were: President, Bernice Cohen; Vice-President, John Euller; Secretary-Treasurer, Jane Noller.

TWO RECTANGLES IN A QUARTER-CIRCLE

B. M. STEWART, Michigan State College

1. The problem. Consider the rectangle A, bounded by $x=0$, $x=\cos a$, $y=0$, $y=\sin a$, and of area $A=\cos a \sin a$; and the rectangle B, bounded by $x=0$, $x=\cos b$, $y=\sin a$, $y=\sin b$, and of area $B=\cos b\,(\sin b-\sin a)$; the angles, a and b, being subject to the restriction $0<a<b<90°$. (See Figure 1.) The problem is to maximize the function $z=A+B$, subject to the auxiliary condition $A=B$.

2. The interest of the problem. A substitution shows $z=2A=\sin 2a$, so that at first sight the solution seems to be $a=45°$, but the restriction on the companion angle b—which must also be in the first quadrant—makes this solution not acceptable. Here then is a situation of interest to students of the calculus, for the ordinary method of finding a maximum has failed because of a restriction on the range of the variable.

3. The source of the problem. Furthermore, the restriction is not artificial, for this problem arises in electrical engineering in the design of a winding-core with the figure described above representing one-fourth of the total cross section. (See Figure 2.)

4. General methods of solution. The student with broad interests will want to investigate the general methods for solving problems of the type here encountered—namely, to maximize the function $z=f(a, b)$ subject to the auxiliary condition $g(a, b)=0$. These general methods are presented in advanced calculus texts under the topics of partial differentiation, or differential geometry, or Lagrange's method of multipliers. Even with these methods caution must be used if there are restrictions imposed on the range of the variables.

5. A solution by elementary calculus. But this problem is of a special character. It is clear from the figure that if angle a is between 45° and 90°, then area B cannot be as great as area A. Since the function $z=\sin 2a$ is increasing for $0<a<45°$, the problem is to find the greatest a in the range $0<a<45°$ for which the equation $A=B$ has a solution b satisfying $0<a<b<90°$. Conceivably this may be solved by treating the relation $A=B$ as in elementary calculus for a maximum of a—namely, by setting $da/db=0$. The following equations are obtained:

(1) $A = B$; $\quad \cos a \sin a = \cos b \sin b - \cos b \sin a$;

(2) $da/db = 0$; $\quad \cos^2 b - \sin^2 b + \sin a \sin b = 0$.

By squaring and by using simple trigonometric identities it is possible to eliminate b from equations (1) and (2) and to obtain the following equation in which $Q=\sin^2 a$:

(3) $$(Q-1)(15Q^3+3Q^2+10Q-1)=0.$$

From equation (3) one and only one value of a, with an acceptable companion b, can be obtained. The approximate solution of the problem is as follows:

$$a = 18^\circ\ 2'\ 27'', \qquad b = 52^\circ\ 4'\ 10''.$$

The meaning of this solution is clarified by constructing the graph of the relation $A = B$ (see Figure 3), a task which is much simplified by noticing the symmetry in the points (0, 0) and (90°, 90°) and the symmetry in the line $b = a + 90^\circ$.

6. Another solution by elementary calculus. The following method is indirect, yet worthy of a brief description. For each fixed choice of angle a there can be found by solving $dB/db = 0$ just one value of b such that $0 < a < b < 90^\circ$ and such that B has a maximum value, say $M(a)$, so designated because it depends upon a. It is evident from the figure that for small values of a the value of $M(a)$ exceeds A, hence it is possible to solve the equation $A = B$; whereas for large values of a the value of $M(a)$ is less than A, hence it is impossible to solve $A = B$. The equation $M(a) = A$ determines the boundary between these cases and determines the largest value of a for which the equation $A = B$ has a solution with acceptable values of a and b. The equation $M(a) = A$ is equivalent to equation (3). Since the solution $a = 18^\circ\ 2'\ 27''$ is in the range $0 < a < 45^\circ$ where s is increasing, this maximum a serves to maximize s under the auxiliary condition $A = B$.

7. A solution without calculus. The maximum a with a companion b satisfying $A = B$ and $0 < a < b < 90^\circ$ may also be determined by using the theory of equations.

Consider the rectangles in the figure. For small angles a there are two choices of b such that $A = B$. For large angles a there are no choices of b. Between these cases is a certain angle a for which there is one choice of b. This situation may be studied algebraically by replacing the equation (1) after some rearrangement, squaring, and use of identities by the following equation which may be described as a reduced quartic in the variable $\cos b$ with coefficients involving $\cos a$:

$$(4)\quad \cos^4 b - \cos^2 a \cos^2 b + 2(\cos a - \cos^3 a)\cos b + (\cos^2 a - \cos^4 a) = 0.$$

First, Descartes' rule of signs shows there are two or no real positive roots, confirming the geometric intuition used above. Secondly, the inbetween case of two real and positive but identical roots can be determined by the vanishing of the discriminant of the quartic. The computation of the discriminant for equation (4) is not an easy matter, but after considerable simplification the discriminant may be written in the following way with $P = \cos^2 a$:

$$-16P^2(P-1)(15P^3 - 48P^2 + 61P - 27).$$

Under the substitution $P = \cos^2 a = 1 - \sin^2 a = 1 - Q$ the discriminant takes the form which follows:

$$-16Q(1-Q)^2(15Q^3 + 3Q^2 + 10Q - 1).$$

Setting the discriminant equal to zero leads to but one acceptable solution, in agreement with that obtained from equation (3).

8. A trigonometric oddity. In the equation $A=B$ one of the two solutions for b, when $a=18°$, is $b=54°$, exactly! The proof of this fact is an interesting exercise in trigonometry. The second solution of $A=B$, when $a=18°$, is given by $b=50°\ 7'$, approximately.

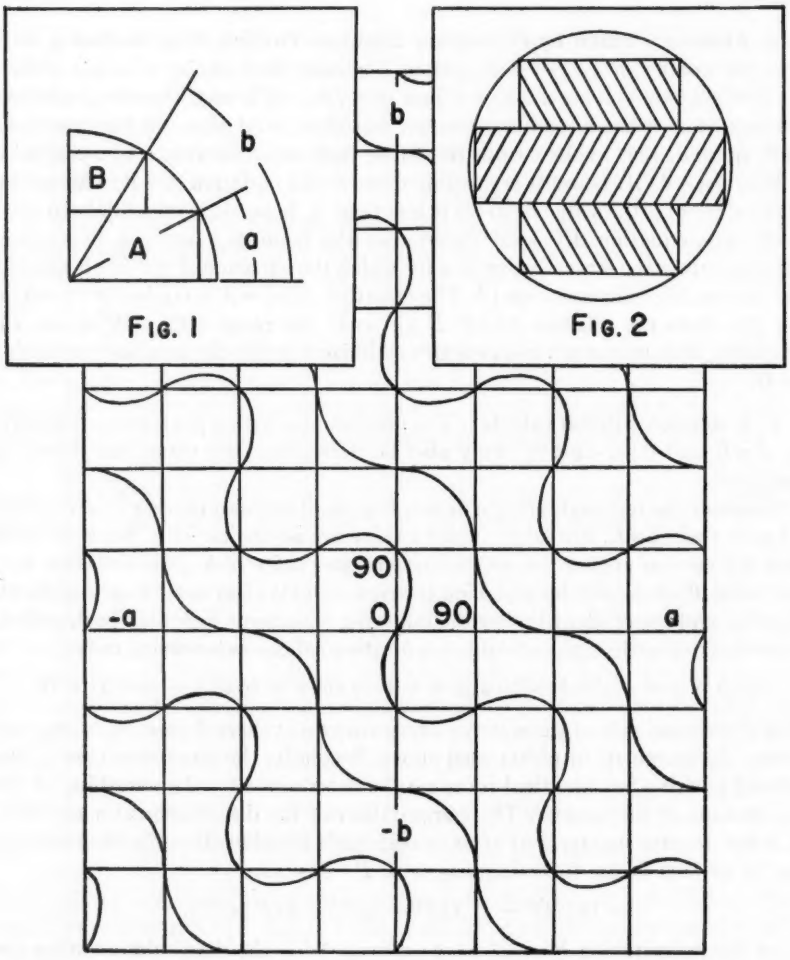


FIG. 1

FIG. 2

FIG. 3

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 656. *Proposed by S. H. Gould, National Research Council, Ottawa*

Each member of a very large office-staff agrees to buy a fifty-dollar war bond. One of the members suggests that each bond, instead of being given to its purchaser, be disposed of by lot. He wishes to wager one dollar that he himself will suffer loss through his own suggestion. How much should be wagered against him?

E 657. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the locus of centers of spheres passing through two given points and touching a given sphere.

E 658. *Proposed by Norman Anning, University of Michigan*

Find three three-digit numbers in geometrical progression which can be derived from one another by cyclic permutation of digits.

E 659. *Proposed by R. A. Staal, University of Toronto*

Show that, if one conic is self-reciprocal with respect to another, then the two conics belong to a symmetrical set of four, each of which is self-reciprocal with respect to any of the other three. (However, not more than three of the four conics can be real.)

E 660. *Proposed by C. D. Olds, Purdue University*

Let $z_0, z_1, \cdots, z_k$ be $k+1$ different complex numbers, all contained in the circle $|z| \leq r$. Let

$$B_{kp} = \begin{vmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^{k-1} & z_0^{k+p} \\ 1 & z_1 & z_1^2 & \cdots & z_1^{k-1} & z_1^{k+p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z_k & z_k^2 & & z_k^{k-1} & z_k^{k+p} \end{vmatrix}.$$

Prove that

$$\left| \frac{B_{kp}}{B_{k0}} \right| \leq \binom{k+p}{p} r^p.$$

SOLUTIONS

Sums of Products of Digits

E 622 [1944, 285]. *Proposed by V. Thébault, Tenne, Sarthe, France*

Find the smallest four-digit number such that the sum of products of pairs of digits is equal to the sum of products of sets of three.

Solution by Frank Hawthorne, New Rochelle, N. Y. Trying $a=1$ in the equation $ab+ac+ad+bc+bd+cd=abc+abd+acd+bcd$, we obtain

$$b + c + d = bcd.$$

If we exclude the trivial case $b=c=d=0$, the smallest solution is 1, 2, 3, giving the desired number 1123.

Also solved by Murray Barbour, D. H. Browne, W. E. Buker, H. N. Carleton, M. I. Chernofsky, E. D. Schell, E. P. Starke, W. R. Talbot, Hazel Schoonmaker Wilson, and the proposer. Mrs. Wilson remarks that, if duplicate digits are not allowed, the number is 2036.

Homothetic Tetrahedra and Coaxal Spheres

E 623 [1944, 285]. *Proposed by N. A. Court, University of Oklahoma*

The circumsphere of a tetrahedron $ABCD$ meets four "cevians" LA, LB, LC, LD in the points A', B', C', D'; $A''B''C''D''$ is a tetrahedron homothetic to the tetrahedron $A'B'C'D'$ with respect to the point L. If P and Q are two points in space, show that the four spheres $AA''PQ$, $BB''PQ$, $CC''PQ$, $DD''PQ$ are coaxal.

Solution by the Proposer. We have

$$LA \cdot LA' = LB \cdot LB' = LC \cdot LC' = LD \cdot LD' = p,$$

where p is the power of the point L for the sphere $ABCD$. Again, we have

$$LA'' = kLA', \quad LB'' = kLB', \quad LC'' = kLC', \quad LD'' = kLD',$$

where k is the value of the homothetic ratio. Hence

$$LA \cdot LA'' = LB \cdot LB'' = LC \cdot LC'' = LD \cdot LD'' = pk.$$

Thus the point L has equal powers with respect to the four spheres $AA''PQ$, $BB''PQ$, $CC''PQ$, $DD''PQ$, and LPQ is the radical plane of any two of these four spheres. This proves the proposition.

A Property of Subfactorials

E 624 [1944, 285]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the integer nearest to $n!/e$ is a multiple of $n-1$.

Solution by Irving Kaplansky, Columbia University. The error made in stopping the expansion of e^{-1} with the term $(-1)^n/n!$ is less than $1/(n+1)!$. Hence the integer nearest to $n!/e$ is

$$P_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right\} = \Delta^n 0!,$$

the well known solution of the *Problème des Rencontres*. The divisibility property in question can be verified as follows:

$$\begin{aligned} P_n &= nP_{n-1} + (-1)^n = (n-1)P_{n-1} + P_{n-1} + (-1)^n \\ &= (n-1)P_{n-1} + (n-1)P_{n-2} + (-1)^{n-1} + (-1)^n \\ &= (n-1)(P_{n-1} + P_{n-2}). \end{aligned}$$

Also solved by Murray Barbour, J. B. Kelly, L. M. Kelly, Norman Miller, E. D. Schell, E. P. Starke, and the proposer.

An Extension of E 481

E 625 [1944, 285]. *Proposed by C. A. B. Smith, Trinity College, Cambridge, England*

Let X_i, Y_i, Z_i be the cofactors of x_i, y_i, z_i in the general third-order determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = D.$$

Prove that

$$\begin{vmatrix} X_2X_3 & Y_2Y_3 & Z_2Z_3 \\ X_3X_1 & Y_3Y_1 & Z_3Z_1 \\ X_1X_2 & Y_1Y_2 & Z_1Z_2 \end{vmatrix} = -D^2 \begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}.$$

Solution by C. V. L. Smith, P. G. School, U. S. Naval Academy. Multiplying the respective rows of the determinant

$$\begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}$$

by x_1, x_2, x_3, and then subtracting the first row from the second and third, we find that it is equal to

$$\frac{1}{x_1x_2x_3} \begin{vmatrix} x_1x_2x_3 & x_1y_2y_3 & x_1z_2z_3 \\ 0 & -y_3Z_3 & z_3Y_3 \\ 0 & y_2Z_2 & -z_2Y_2 \end{vmatrix} = y_2z_3Y_2Z_3 - y_2z_3Y_3Z_2.$$

By a well-known result on determinants (see, *e.g.*, Bôcher, *Introduction to Higher Algebra*, p. 33, Cor. 1), the cofactors of the terms of the adjoint of D are equal to the corresponding terms of D multiplied by the value of D. Hence from the result already obtained it follows that

$$\begin{vmatrix} X_2X_3 & Y_2Y_3 & Z_2Z_3 \\ X_3X_1 & Y_3Y_1 & Z_3Z_1 \\ X_1X_2 & Y_1Y_2 & Z_1Z_2 \end{vmatrix} = D^2(Y_3Z_2y_2z_3 - Y_2Z_3y_3z_2)$$

$$= -D^2 \begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}.$$

Also solved by A. R. Stokes.

The Position of a Ship

E 626 [1944, 347]. *Proposed by W. E. Buker, Perry High School, Pittsburgh*

A ship at sea attempts to determine its position by plotting lines of position through observation of three stars. Due to inaccuracy of observation, these lines are not concurrent but form a triangle. What is the most probable position of the ship, assuming (a) that the errors are due to defective instruments and are of the same kind, (b) that the errors are due to inaccurate observation and may not all be of the same kind?

Solution by R. H. Wilson, Jr., U. S. Naval Academy. The three lines of position represent the observed bearings of three objects whose positions, with reference to the coordinates of the chart, are known. (Such definition would then include landmarks as well as stars; but we shall assume that, as in the case of stars, parallactic differences are negligible.) Consider the probable bearing of the ship as viewed from these exterior points. It must lie toward the interior of the triangle as viewed from at least two of the points, if the angular value of the correction is to be arithmetically equal for all three. This requires that at least two errors must be of the "same kind."

(a) When all three errors are of the same kind, the true position must lie within the triangle, at such a point that the angular corrections are equal. Thus the most probable position of the ship is a point equidistant from the three sides of the triangle, namely the *incenter*.

(b) When the error in any one bearing may be of the opposite kind, the most probable position will be a point toward the exterior of the triangle as viewed from the object on which the error of the opposite kind was made. Again assuming equality of absolute values of all errors, the probable point will be an excenter, *viz.*, the center of the escribed circle exterior to the side on which this error of the opposite kind was made. Assuming equal probability for an error of the opposite kind on any one of the three bearings or on none at all, the three *excenters* and the *incenter* are positions of equal weight. The most probable position of the ship is then the mean position of these four points.

The resulting probable position is, in many cases, surprisingly far from the incenter (the point usually assumed by navigators), especially when one angle is obtuse. However, practical navigation will include simultaneously both conditions (a) and (b); so the position of the incenter should be weighted more heavily. This discrepancy partly explains the navigators' traditional prejudice in favor of an equilateral triangle.

Also solved by R. A. Rosenbaum, who remarks that a complete discussion is contained in an article entitled "Accurate Determination of the Position at Sea," in the *Hydrographic Review* for Nov., 1931.

An Enneagon

E 627 [1944, 347]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct an enneagon, given the centers of exterior squares described on the sides.

Solution by Robert Steinberg, University of Toronto. Suppose the nine given centers are $A_{01}, A_{12}, \cdots, A_{80}$. Then we will have solved the problem when we have found one vertex, A_0, of the enneagon $A_0A_1 \cdots A_8$. Start with an arbitrary point P_0 in the plane; join it to A_{01} and mark off $A_{01}P_1 = A_{01}P_0$ on the perpendicular to $A_{01}P_0$ at A_{01}, so that $\sphericalangle P_0A_{01}P_1$ is one right angle in the positive sense. Next, join P_1 to A_{12} and mark off $A_{12}P_2 = A_{12}P_1$ on the perpendicular to $A_{12}P_1$ at A_{12}, so that $\sphericalangle P_1A_{12}P_2$ is one right angle. Continuing in this way, we get a sequence of points $P_0, P_1, \cdots, P_8, P_9 = P'$ (say). If P' coincides with P_0, we have finished, and $P_0P_1 \cdots P_8$ is the required enneagon. If not, let Q_0 be a point different from P_0. Going through the same construction as before, we get a sequence of points $Q_0, Q_1, \cdots, Q_8, Q'$. If we can determine Q_0 so that Q' coincides with Q_0, then this will be our required vertex.

Now, let us regard $P_0Q_0, P_1Q_1, \cdots, P'Q'$ as different positions of a directed segment PQ, and let us see what happens to this segment in moving from P_0Q_0 to P_1Q_1, and so on. Since $A_{01}P_0Q_0$ and $A_{01}P_1Q_1$ are congruent triangles, we have $P_1Q_1 = P_0Q_0$. Also $\sphericalangle(P_0Q_0, P_1Q_1)$ is a positive right angle. Thus, in going from P_0Q_0 to P_1Q_1, PQ has remained the same in length and has turned through one right angle. Hence, in going from P_0Q_0 to P_9Q_9 or $P'Q'$, PQ remains the same in length and is turned through nine right angles, or effectively one right angle. If $Q' = Q_0$, we must have $P_0Q_0 = P'Q_0$, and $\sphericalangle P_0Q_0P'$ equal to one right angle. This determines $Q_0 = A_0$ uniquely as the third vertex of one of the isosceles right-angled triangles standing on P_0P' as hypotenuse, namely the one for which $\sphericalangle P_0A_0P'$ is a positive right angle.

We can extend this solution to the general case of an n-gon; but we have to distinguish four cases, depending on the residue of n modulo 4. The above solution typifies the case when $n \equiv 1 \pmod 4$. If $n \equiv 3 \pmod 4$, PQ will be turned through an angle congruent to three right angles, and the vertex A_0 will be uniquely determined as the third vertex of the *other* isosceles right-angled triangle standing on P_0P' as hypotenuse, namely the one for which $\sphericalangle P_0A_0P'$ is a negative right angle. If $n \equiv 2 \pmod 4$, PQ will be turned through an angle congruent to two right angles, *i.e.*, reversed in direction, and the vertex A_0 will be uniquely determined as the midpoint of P_0P'. Finally, if $n \equiv 0 \pmod 4$, PQ will undergo several complete turns, making $P'Q'$ parallel and equal to P_0Q_0. Then $Q' = Q_0$ if and only if $P' = P_0$; so we have either no solution or else an infinite number of solutions (with A_0 an arbitrary point in the plane). To sum up, the solution is unique or poristic according as $n \not\equiv 0$ or $n \equiv 0 \pmod 4$.

Our construction can readily be extended to solve the more general problem of constructing a polygon $A_0A_1 \cdots A_{n-1}$ such that the triangles $A_0A_{01}A_1$, $A_1A_{12}A_2$, $\cdots$, shall be similar to n given triangles, where A_{01}, A_{12}, $\cdots$ are n given points. A particular case of this more general problem is mentioned in the *Editorial Note* to 4076 [1944, 411].

Also solved by Howard Eves, J. B. Kelly, Joseph Rosenbaum, and the proposer.

A Highly Composite Number

E 628 [1944, 348]. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

Find the smallest positive integer, one-half of which is a square, one-third of which is a cube, and one-fifth of which is a fifth power.

Solution by C. B. Barker, Jr., University of New Mexico. Clearly, the desired number is of the form $N = 2^p3^q5^r$.

Since $N/2$ is a square, $q \equiv r \equiv 0 \pmod 2$ and $p \equiv 1 \pmod 2$.

Since $N/3$ is a cube, $p \equiv r \equiv 0 \pmod 3$ and $q \equiv 1 \pmod 3$.

Since $N/5$ is a fifth power, $p \equiv q \equiv 0 \pmod 5$ and $r \equiv 1 \pmod 5$.

The smallest values of p, q, r satisfying these conditions are

$$p = 15, \quad q = 10, \quad r = 6.$$

So the smallest value of N is

$$N = 2^{15}3^{10}5^6 = 30{,}233{,}088{,}000{,}000.$$

Also solved by Murray Barbour, Shepard Bartnoff, H. W. Becker, R. G. Blake, Colin Blyth, D. H. Browne, W. E. Buker, L. H. Bunyan, H. N. Carleton, F. M. Carpenter, Olive E. Cory, J. S. Cromelin, Monte Dernham, C. W. Emmons, Howard Eves, Daniel Finkel, Clifford Gardner, George Grossman, R. W. Hamming, Frank Hawthorne, Irving Kaplansky, Norbert Kaufman, J. B. Kelly, R. T. Luginbuhl, G. W. Petrie, E. D. Schell, E. P. Starke, W. R. Talbot, Hazel Schoonmaker Wilson, R. H. Wilson, Jr., H. J. Zimmerberg, and the proposer. Kaplansky remarks that, if it is further required that one-seventh of N be a seventh power, then the smallest value of N is

$$2^{105}3^{70}5^{126}7^{120},$$

a number with 255 digits.

A Finite Series

E 629 [1944, 348]. *Proposed by H. S. M. Coxeter, University of Toronto*

Sum the series

$$\sum_{r=0}^{[n/2]} (-\tfrac{1}{4})^r \binom{n-r}{r}.$$

Solution by G. W. Petrie, USNR Midshipmen's School, Notre Dame, Ind. Let the sum of the given series be S_n. Since

$$\binom{n-r}{r} = \binom{n-r-1}{r} + \binom{n-r-1}{r-1},$$

we find

$$S_n = S_{n-1} - \tfrac{1}{4}S_{n-2}. \tag{1}$$

This may be simplified by the substitution $S_n = U_n/2^n$, which gives

$$U_n = 2U_{n-1} - U_{n-2},$$

whence

$$U_n - U_{n-1} = U_{n-1} - U_{n-2} = \cdots = U_1 - U_0 = 2S_1 - S_0 = 2 - 1 = 1,$$

$$U_n - U_0 = n,$$

$$U_n = n + 1,$$

$$S_n = (n+1)/2^n.$$

Also solved by H. W. Becker, C. S. Gardner, Irving Kaplansky, J. B. Kelly, Norman Miller, and M. Wyman. D. H. Browne remarks that, for the series with 1 instead of $-\frac{1}{4}$, (1) becomes

$$S_n = S_{n-1} + S_{n-2};$$

hence the expression

$$\sum_{r=0}^{[n/2]} \binom{n-r}{r}$$

for Fibonacci numbers.

Editorial Note. In terms of $U_n(x)$, the Tschebyscheff polynomial of the second kind, we have

$$2^n S_n = U_n(1) = n + 1.$$

This problem occurs as "Ex. I" in Lucas, *Théorie des Nombres*, vol. 1 (Paris, 1891), p. 464. It can be derived from Bernoulli's expansion

$$\frac{\sin(n+1)\theta}{\sin\theta} = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} (2\cos\theta)^{n-2r}$$

by making θ tend to zero. Bernoulli's expansion itself may be verified by means of the recurrence formula

$$U_n = 2\cos\theta U_{n-1} - U_{n-2},$$

or obtained directly by equating coefficients of t^n in

$$2\sum_{n=1}^{\infty} \frac{t^n \cos n\theta}{n} = -\log(1 - 2t\cos\theta + t^2) = \sum_{k=1}^{\infty} \frac{(2t\cos\theta - t^2)^k}{k}$$

and then differentiating with respect to θ.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4148. *Proposed by B. M. Stewart, Michigan State College*

A variation of the game of Solitaire is this game of the Red Cross, which is played with checkers on a field of squares arranged in six rows and five columns. During the play there can be at most one piece on a square, and at the outset one square is free so that the play, which consists of a series of jumps, may begin. A jump may be made if in three consecutive squares A, B, C of any row or column there are pieces on A and B while C is free. The jump is moving the piece on A over B to C and removing the piece on B from the play. Jumps along the diagonals of the field are forbidden. The object of the play is to remove all the pieces from the field except five, which are to be left in the form of the Red Cross, as in figure 1.

By the known theory for Solitaire (for example, see this MONTHLY, April, 1941, p. 228) a necessary condition is that the initially empty square be either as in figure 2 or 3. Show that in each of these cases a solution is possible.

x x x x x	0 0 0 0 0	0 0 0 0 0
x x 0 x x	0 0 0 0 0	0 0 0 0 0
x 0 0 0 x	0 0 x 0 0	0 0 0 0 0
x x 0 x x	0 0 0 0 0	0 0 0 0 0
x x x x x	0 0 0 0 0	0 0 0 0 0
x x x x x	0 0 0 0 0	0 0 x 0 0
Figure 1	Figure 2	Figure 3

4149. *Proposed by N. A. Court, University of Oklahoma*

The powers of the vertices of a tetrahedron (T) with respect to the spheres determined by the centroid of (T) and the circles of intersection of the respectively opposite faces of (T) with a sphere (L), of arbitrary radius, concentric with the circumsphere of (T), are equal.

4150. *Proposed by V. Thébault, Tenne, Sarthe, France*

In a tetrahedron $ABCD$ with the orthocenter H, the perpendiculars at A to the faces ACD, ABD, ABC meet respectively the planes HCD, HBD, HBC in (A_1, A_2, A_3), and similarly for the points (B_1, B_2, B_3) *etc.* Show that the planes $A_1A_2A_3$, $B_1B_2B_3$, *etc.*, are perpendicular to the medians of $ABCD$.

SOLUTIONS

A Special Triangle Transversal

4102 [1943, 638]. *Proposed by Hüseyin Demir, Columbia University*

Let O and I be respectively the circumcenter and incenter of a given triangle ABC. Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l. Prove synthetically that: (1) The points A_0, B_0, C_0 are collinear; (2) The sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l; (3) the line $A_0B_0C_0$ is perpendicular to the line OI.

Solution by the Proposer. (1) The locus of points whose sum of distances to the sides CA, AB is l, is a straight line passing through A_0, and perpendicular to AI. Let B_c, C_b be points where this locus cuts CA, AB. Similarly we consider two other loci corresponding to B_0, C_0. Let $A'B'C'$ be the triangle formed by these three loci. We shall prove that the last triangle is in perspective with ABC, I being the center of perspective. This is obvious, because since A' is the intersection of two loci, its distances to CA, AB are equal, that is, A' belongs to AI. Similarly B', C' belong respectively to BI, CI. Thus applying Desargue's theorem we have collinearity of A_0, B_0, C_0.

(2) Let M be a point of $A_0B_0C_0$ with x, y, z its distances to BC, CA, AB. We shall prove that $x+y+z=l$. Consider the locus of points with $y+z=Cst.$ This locus MQ (see figure) is parallel to B_cC_b, and $QQ_1=y+z$. Now, C_b, A_b having equal distances l to CA (see (1)) what we have to prove is that $QQ_2=x$. Draw MP parallel to BC, then $x=MX=PP_1$. Since $A'A_b$ is the bisector of $A_0A_bC_b$, we have $x=PP_1=PP_2$. It remains to prove that $QP \| C_bA_b$. This is true because the two triangles QMP, $C_bA_0A_b$ have two sides parallel, namely QM, C_bA_0 and MP, A_0A_b and they are in perspective, with C_0 as center of perspective. Therefore their third sides QP, C_bA_b must be parallel, that is $x=PP_2=QQ_2$.

(3) We shall prove two things: (a)—$A_0B_0C_0$ is the radical axis of circles (ABC) and $(A'B'C')$. (b)—The center O' of $(A'B'C')$ lies on OI, thus property (2) will be proved.

(a)—For, observe the relation $\overline{C_0A}\cdot\overline{C_0B}=\overline{C_0A'}\cdot\overline{C_0B'}$. This is true because the quadrilateral $ABB'A'$ is cyclic. (Note the equality of angles $A_bB'B=A'AB=\frac{1}{2}A$). Thus C_0 has equal powers with respect to the two circles. A similar property holds for A_0, B_0.

(b)—To prove that O' belongs to OI we shall remark that the locus of O' is a straight line when $A'B'C'$, whose sides are perpendicular to AI, BI, CI, varies, and since $A'B'C'$ is always in perspective with ABC, with I the center of perspective, O' will describe a straight line passing through I. It also passes through O. For, let A' be taken at the point where AI meets the circle (ABC). It is easy to see that B', C' will be similar points on the same circle. Thus O', the center of $(A'B'C')$, coincides with O, the center of (ABC). Therefore the radical axis $A_0B_0C_0$ of (ABC) and $(A'B'C')$ is perpendicular to the line OI passing through the centers O and O'.

Editorial Note. The first two theorems follow from similar triangles. The

case of an isosceles ABC may be discarded. For, if say the sides AB, AC have equal lengths, then in consequence of symmetry about AI the same is true for AB_0, AC_0, B_0 and C_0 being respectively on AC and AB. It then follows that B_0C_0 is perpendicular to OI; the converse is true as well as parts (1) and (2), but A_0 has an exceptional position. The points A_0, B_0, C_0 are uniquely determined by the given constant l. The distances x_b, y_b, z_b for B_0 are such that $y_b=0$, $x_b+z_b=l$, *etc.* Let P be a point on the straight line of C_0B_0 and let it divide this segment in the ratio $\lambda:1$. Then we have

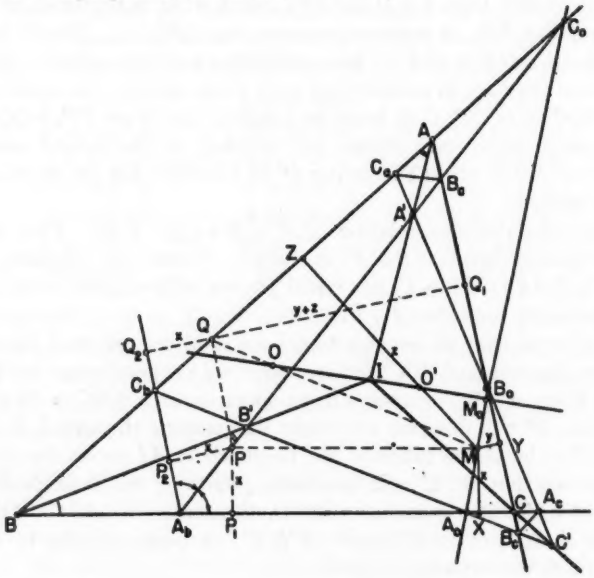
$$(\lambda+1)x = x_c + \lambda x_b, \qquad (\lambda+1)y = y_c, \qquad (\lambda+1)z = \lambda z_b,$$

where x, y, z are the distances for P. By addition we have

$$(\lambda+1)(x+y+z) = (x_c+y_c) + \lambda(x_b+z_b) = (\lambda+1)l;$$

and, if P is a finite point $x+y+z=l$. The straight line C_0B_0 meets BC in a finite point for which $x_a=0$ and $y_a+z_a=l$; hence this point is A_0. The two straight lines $A_0B_0C_0$ for different values of l are parallel; for, if they meet in a finite point, this point would have the sum of its distances equal to two different values. If P is a point not on B_0C_0 the line through it parallel to the latter meets the two sides in points different from B_0 and C_0. Hence the sum of its distances must be different from l; and this proves that the locus of points for a given l is the straight line $A_0B_0C_0$ for that value of l.

In the special case where $A'B'C'$ is inscribed in (O) the polar of C_0 passes through $(AA', BB')=I$, similarly, the polar of B_0 passes through I. Hence the polar B_0C_0 of I is perpendicular to OI.



NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. E. G. Begle of Yale University has been promoted to an assistant professorship.

Assistant Professor H. R. Branson of Howard University has been promoted to an associate professorship.

Associate Professor W. H. Brothers of Talladega College, Talladega, Alabama, has been promoted to a professorship.

J. O. Brown of Hampton Institute, Hampton, Virginia, has been promoted to an assistant professorship of physics.

Assistant Professor R. E. Byrne and Associate Professor A. E. Taylor of the University of California, Los Angeles, have been granted leaves of absence, the former for war research work and the latter to serve as operational research consultant with the Army Air Forces in England.

Dr. E. W. Cannon of the University of Delaware has been granted leave of absence for military service with the rank of Lieutenant Commander at Philadelphia Navy Yard.

Sister M. Loyola Conlan of the College of Mt. St. Vincent, New York City, has been promoted to a professorship and the acting chairmanship of the department.

Associate Professor C. M. Cramlet of the University of Washington has been granted leave of absence to serve with the Twenty-first Bomber Command operating in the Pacific.

Professor A. R. Crathorne of the University of Illinois has retired with the title professor emeritus.

R. E. Fullerton of Yale University has been appointed to an assistant professorship at the University of Oklahoma.

Dr. B. E. Gillam of the University of Missouri has been appointed to an assistant professorship at Drake University, Des Moines, Iowa.

Associate Professor Mary Goins of Western College has been appointed to an associate professorship at DePauw University.

F. F. Helton of the University of Illinois has been appointed to an associate professorship at Central College, Fayette, Missouri.

Dr. A. W. Jones of Yale University has been appointed to an assistant professorship at Michigan State College.

Assistant Professor J. F. Kubis of Fordham University has been promoted to an associate professorship.

Assistant Professor J. N. McClelland of Drake University is now engaged in war work at California Institute of Technology.

Professor W. H. McCrea of Queen's University, Belfast, Northern Ireland, has been appointed to a professorship at the University of London.

Associate Professor Florence M. Mears of George Washington University has been promoted to a professorship.

Associate Professor Cronan Mullen of Siena College, Loudonville, New York, has been promoted to a professorship in physics and the acting chairmanship of the Science Department.

Dr. G. M. Robison of Susquehanna University has been promoted to an assistant professorship.

Assistant Professor A. E. Ross of St. Louis University has been promoted to an associate professorship.

Dr. T. H. Southard of Wayne University has been promoted to an assistant professorship.

Assistant Professor W. G. Warnock of the University of Alabama has been promoted to an associate professorship.

Associate Professor S. S. Wilks of Princeton University has been promoted to a professorship.

J. F. Wyckoff of Trinity College has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
Oregon State College: Dr. S. P. Avann
Queens College: Dr. Ruth O. Goodman, Dr. J. C. R. Li
University of Michigan: Dr. O. G. Owens
University of Saskatchewan: Dr. W. J. R. Crosby

Professor Paul Capron, formerly of the United States Naval Academy, died September 29, 1944. He was a charter member of the Mathematical Association.

Dr. R. R. Fleet, formerly head of the mathematics department of William Jewell College, Liberty, Missouri, died December 1, 1944. He was a charter member of the Mathematical Association.

Lieutenant F. E. Riley, Jr., formerly a teacher in North Phoenix High School, Arizona, was killed in action in the China Burma India theater October 22, 1944.

Professor Emeritus C. L. Thornburg of Lehigh University died October 14, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE ESMWT PROGRAM OF PENNSYLVANIA STATE COLLEGE

The ESMWT Program supervised by the Pennsylvania State College has been of considerable interest to educators because of its extent and several novel experiments in organization. The following brief report treats those features of the Program of especial interest to mathematicians.

The Pennsylvania State College has offered work under ESMWT in all departments of engineering, in a number of branches of the mineral industries, and in chemistry, physics, mathematics, and economics. Some of the work has been done in residence, but most of it has been carried on by extension in more than a hundred communities throughout Pennsylvania. All extension classes are supervised by members of the regular residence departments of the College. A separate operating division has handled the business details of ESMWT, together with a number of regular extension activities. This central plan of organization has been quite effective, and has been particularly helpful in the case of courses, such as the one on quality control, which required the coordination of facilities to meet the needs of several departments.

Many courses are offered in communities where the registration is drawn chiefly from several local industries. In communities where large industries are located, registration is frequently limited to the employees of a single company. This is especially true of certain of the more specialized and advanced courses.

The standard courses offered in mathematics start with elementary algebra and trigonometry, and continue through calculus and differential equations. A course in higher mathematics upon the level of advanced calculus and another in the theory of functions of a complex variable have been given at the request of particular engineering groups. In general, however, most of the demand has been for classes on an elementary level.

Several courses have been given which draw their subject matter from two fields. For instance, an introductory course in physics and mathematics has been offered under the title "Foundations of Engineering, I." This course has been requested most frequently of the varous combinations, and interest in it has been at a high level ever since its introduction in 1941; at times thirty or forty sections have been running simultaneously. Laboratory equipment has always been provided for the physics portion of the work, and this has added considerable interest to the project. Mathematics has also been combined with engineering mechanics, with electrical engineering, with radio, and with drafting. Each of these combinations serves a specific type of industrial group; con-

sequently demand for these courses has been less than for the general mathematics and physics course. A combination course in quality control was set up cooperatively by the departments of mathematics and industrial engineering. The course is generally regarded as satisfactory, but requests for it have been limited to those companies which have already made some application of quality control in their inspection procedures. The cooperation of departments in the writing of an outline for a single course has appreciably increased the mutual understanding of the problems of the several departments.

There has been little attempt as yet to evaluate the results of combination courses from such points of view as efficiency of teaching, concentration of emphasis upon essentials of course content, and student interest. However, there appears to be some basis for the belief that combined courses can be used to advantage in the usual college curriculum as well as in the college extension program.

THE NAVY CAMPAIGN FOR RADAR TECHNICIANS

In recent months the Navy has been conducting an intensive campaign to obtain recruits who will study to become radio technicians, with the emphasis upon radar. Voluntary enlistments are open to men of 17 and to those who are 38 and over. Each inductee who passes his pre-induction physical examination and who meets Navy standards may request to take the Eddy Aptitude Test. Those who pass are assured of assignment to the Navy for radio technician training.

No extensive technical background is necessary to pass the Eddy Test. It is essential, however, to have a knowledge of elementary mathematics and physics. Some acquaintance with the principles of radio is also helpful.

Men who pass the Eddy Test become Seamen First Class, and are sent to a Naval Training Center for indoctrination. After that, each man receives nine months of specialized training and a Petty Officer rating. The program may be of interest to young men who have been able to study a term or two of college mathematics and physics before induction.

THE EDUCATIONAL PROGRAM FOR CANADIAN VETERANS

The leaders of Canada in the future must come, in large measure, from those young Canadians who have volunteered for active service in this war. This is a principle which the Dominion has recognized in setting up its plans for post-war rehabilitation of the armed services, with the result that full opportunity to resume education is given to the young ex-serviceman, or woman, whose ambition to go to university was interrupted by the war. Under the Post-Discharge Re-establishment Order of October 1, 1941, a single man can obtain a maintenance grant of $60 per month. Married men receive $80 per month, and, in addition, dependents' allowances may be granted. Moreover, university fees are paid.

With 47 per cent of service personnel in this war having high or technical school training as compared with 13 per cent in the last war, the number of those

who might qualify for university training is large. All who were qualified for university admission at the time of enlistment, or who can qualify within fifteen months after discharge, may receive this opportunity.

In addition to the fact that initial educational standards are higher, the avenue of educational opportunity within the services has been opened more widely through the establishment of Directorates of Education in each of the services and through the opportunities provided by the Canadian Legion Educational Services.

Length of service, in the first instance, is the governing factor in the period of time for which an exserviceman or woman applying for university training may receive Government assistance. Grants are awarded on a month-for-month basis, for the time veterans were in the service, providing progress is satisfactory. For instance, an ex-serviceman with two full years in the armed services could receive twenty-four months of assistance. This would mean that he could complete three academic years (each year figured on eight school months) with Government assistance. However, the Government is of the opinion that the opportunity to complete the requirements for a degree should be available to the outstanding student. If, then, at the conclusion of the period of training to which he is entitled by his war service, a student has demonstrated the necessary ability, the assistance may be extended until he has completed his university course. Moreover, if his scholarship is sufficiently outstanding, he may be assisted in post-graduate studies. Post-graduate opportunities also are available to those who had entered upon, or were about to enter upon post-graduate studies at the time of their enlistment.

The regulations provide that no one may repeat a year's work for which benefits have already been paid, nor may he exhaust benefits in university study and then expect to receive vocational training.

This post-discharge training program is now in effect, and a number of exservicemen and women discharged from the armed forces in this war are attending Canadian universities and normal schools. Under a recent amendment, university education was authorized in exceptional cases at any university or college "of educational standards approved by the Minister" when suitable facilities are not available in Canada.

Provision has also been made for a wide program of vocational training for ex-service personnel. Under the Order, maintenance funds are provided, provisions are made for the selection of trainees, and courses of training are authorized. If ex-service men or women believe that a course of vocational training will assist in their rehabilitation, they first have a preliminary interview with the Veterans' Welfare Officer of the Department of Pensions and National Health stationed in the Employment and Selective Service Office in most of the larger centers of the Dominion. Where there is no Welfare Officer immediately available, local Canadian Legion Branches have taken on the responsibility for these preliminary interviews. If it is felt, as a result of these first conferences, that applicants will benefit by training, they are referred to the training officer or

counsellor for further interview or recommendation. As a result of this interview, the case comes before the District Rehabilitation Board which has the power to approve grants and training, and the ex-serviceman or woman is then granted a suitable course of vocational training if it is found that such training will assist in rehabilitation.

Maintenance grants for those taking vocational training are the same as for those studying in universities. These grants, however, will not be paid in most cases for any period in excess of 52 weeks. In an exceptional case, if the period of service is longer, the grant may be extended sufficiently to enable a veteran to complete a course of training, provided the full training period does not exceed the length of service. Veterans taking treatment in hospitals may be permitted to take approved correspondence courses with fees paid. This privilege may also be extended to those employed, in cases where part time or evening courses are not available.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-eighth annual meeting of the Mathematical Association of America was held at Chicago, Illinois, on Saturday and Sunday, November 25 and 26, 1944, in conjunction with meetings of the American Mathematical Society. About two hundred and thirty persons attended the meetings, including the following one hundred and sixty members of the Association:

L. K. Adkins, State Teachers College, La Crosse, Wis.
V. W. Adkisson, University of Arkansas
A. A. Albert, University of Chicago
E. F. Allen, Oklahoma A. and M. College
Beulah M. Armstrong, University of Illinois
Max Astrachan, Antioch College
H. G. Ayre, Western Illinois State Teachers College

Ruth M. Ballard, Wright Junior College
R. H. Bardell, University of Wisconsin Extension Division
C. F. Barr, University of Wyoming
Walter Bartky, University of Chicago
Grace E. Bates, University of Illinois
Felix Bernstein, New York University
H. R. Beveridge, Monmouth College
S. F. Bibb, Illinois Institute of Technology
H. L. Black, Westminster College
G. A. Bliss, University of Chicago
Henry Blumberg, Ohio State University
M. G. Boyce, Western Reserve University
R. W. Brink, University of Minnesota
E. L. Buell, Northwestern University
R. S. Burington, Case School of Applied Science
Herbert Busemann, Illinois Institute of Technology
W. H. Bussey, University of Minnesota

W. D. Cairns, Oberlin College
C. S. Carlson, St. Olaf College
W. B. Carver, Cornell University
W. B. Caton, Illinois Institute of Technology
E. W. Chittenden, University of Iowa
R. V. Churchill, University of Michigan
Mary D. Clement, Wells College
R. H. Cole, University of Western Ontario
J. J. Corliss, De Paul University
H. M. Cox, University of Nebraska
D. R. Curtiss, Northwestern University

PAUL D'ARCO, De Paul University
JAMES EDGAR DAVIS, University of Illinois
JOHN DECICCO, Illinois Institute of Technology
J. E. DOTTERER, Indiana Central College

PAUL EBERHART, Washburn University
J. P. ESPOSITO, Crane Technical High School, Chicago
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

WILL FELLER, Brown University
EDNA M. FELTGES, Woodrow Wilson Junior College
W. L. FIELDS, Louisville Municipal College
J. V. FINCH, Capt., U. S. Army
N. J. FINE, Lukas-Harold Corporation, Indianapolis
L. R. FORD, Illinois Institute of Technology
J. S. FRAME, Michigan State College

H. M. GEHMAN, University of Buffalo
J. W. GIVENS, JR., Northwestern University
G. D. GORE, Central Y. M. C. A. College, Chicago
L. M. GRAVES, University of Chicago
LAURA Z. GREENE, Washburn University
V. G. GROVE, Michigan State College

D. W. HALL, University of Maryland
P. R. HALMOS, Syracuse University
R. W. HAMMING, University of Illinois
E. G. HARRELL, State Teachers College, Platteville, Wisconsin
W. L. HART, University of Minnesota
J. O. HASSLER, University of Oklahoma
E. D. HELLINGER, Northwestern University
FRITZ HERZOG, Michigan State College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
J. D. HILL, Michigan State College
T. R. HOLLCROFT, Wells College
R. C. HUFFER, Beloit College
H. K. HUGHES, Purdue University
C. C. HURD, U. S. Coast Guard Academy

M. H. INGRAHAM, University of Wisconsin

R. D. JAMES, University of British Columbia
L. S. JOHNSTON, University of Detroit
B. W. JONES, Cornell University

SAMUEL KARLIN, Illinois Institute of Technology
DORA E. KEARNEY, Iowa State Teachers College
KATHARINE B. KEPPLER, Foxcroft School, Middleburg, Virginia
E. C. KIEFER, James Millikin University
H. R. KINGSTON, University of Western Ontario
J. R. KLINE, University of Pennsylvania
L. A. KNOWLER, University of Iowa
J. C. KOKEN, Parks Air College
W. C. KRATHWOHL, Illinois Institute of Technology
W. H. KURZIN, Herzl Junior College

A. E. LAMPEN, Hope College
JOSEPH LANDIN, University of Notre Dame
E. P. LANE, University of Chicago
R. E. LANGER, University of Wisconsin
D. H. LEAVENS, University of Chicago
A. T. LONSETH, Northwestern University

C. C. MACDUFFEE, University of Wisconsin
H. F. MAC NEISH, Brooklyn College
MORRIS MARDEN, University of Wisconsin
W. T. MARTIN, Syracuse University
MARGARET E. MAUCH, Michigan State College
J. R. MAYOR, Southern Illinois Normal University
KARL MENGER, University of Notre Dame
K. W. MILLER, Commonwealth Edison Company, Chicago
H. J. MISER, Lawrence College
W. L. MISER, Vanderbilt University
A. C. MOELLER, Marquette University
G. E. MOORE, University of Illinois
C. W. MORAN, Lane Technical School, Chicago
EUGENIE M. MORENUS, Sweet Briar College
E. J. MOULTON, Northwestern University
J. R. MUSSELMAN, Western Reserve University

A. L. NELSON, Wayne University
C. V. NEWSOM, Oberlin College
IVAN NIVEN, Purdue University
E. P. NORTHROP, University of Chicago
F. S. NOWLAN, University of British Columbia

E. B. OGDEN, Union College
RUFUS OLDENBURGER, Illinois Institute of Technology
J. M. H. OLMSTED, University of Minnesota
F. W. OWENS, Pennsylvania State College

GORDON PALL, McGill University
P. M. PEPPER, University of Notre Dame

GEORGE PIRANIAN, Northwestern University
J. C. POLLEY, Wabash College
ADRIEN POULIOT, Laval University

J. F. RANDOLPH, Oberlin College
W. R. RANSOM, Tufts College
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C. B. READ, University of Wichita
F. A. REIBER, Chicago
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P. R. RIDER, Washington University
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E. H. ROTHE, University of Michigan

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E. G. SWAFFORD, Park College

H. P. THIELMAN, Iowa State College
T. Y. THOMAS, Indiana University
C. J. THORNE, Louisiana State University

HENRY VAN ENGEN, Iowa State Teachers College
J. I. VASS, University of Wisconsin Extension Division
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K. P. WILLIAMS, Indiana University
R. S. WOLFE, Northwestern University
ALICE K. WRIGHT, Southern Illinois Normal University

J. H. ZANT, Oklahoma A. and M. College
OSCAR ZARISKI, Johns Hopkins University
SISTER M. CLAUDIA ZELLER, College of St. Francis

The meetings were held in the auditorium of the Museum of Science and Industry in Jackson Park at 57th Street and South Shore Drive, and the hotel headquarters were at the Hotels Windermere. Lunches were available at a convenient cafeteria in the Museum.

A joint dinner for the two organizations was held at 6:00 p.m. on Saturday at the Hotel Windermere West. President W. M. Whyburn of Texas Technological College acted as toastmaster. He introduced Professor R. E. Langer of the University of Wisconsin who gave a short talk on post-war problems. At the close of his talk he called attention to the loss to American mathematics in the recent death of Professor G. D. Birkhoff of Harvard University, and the guests at the dinner stood for a moment of silent tribute to the memory of this outstanding member of the two mathematical organizations. Professor W. T. Martin of Syracuse University offered resolutions expressing thanks to Major L. R. Lohr, Director of the Museum of Science and Industry, to Mrs. Florence Carroll of the staff of the Museum, to the management of the Hotels Windermere, and to the Committee on Arrangements, for their cooperation and help in making the meetings pleasant and successful; and these resolutions were adopted by a rising vote.

The sessions of the American Mathematical Society were held on Friday and on Saturday morning and afternoon. On Friday at 7:45 p.m. the eighteenth

Josiah Willard Gibbs Lecture was given by Professor John von Neumann, the title being "The ergodic theorem and statistical mechanics." On Saturday at 2:00 p.m., by invitation of the Program Committee, Professor Will Feller gave an address on "Limit theorems in the theory of probability."

The Mathematical Association held sessions Saturday evening and Sunday morning, the program having been prepared by a committee consisting of L. M. Graves, *Chairman*; L. M. Blumenthal, E. R. Smith. The program follows.

First Session of the Association

Symposium on undergraduate mathematical curricula.

"The changing curriculum in secondary schools and the preparation of teachers," by Professor Raleigh Schorling, University of Michigan.

"The integration of undergraduate mathematics and physics," by Professor Michael Ference, University of Chicago.

"Applied courses for students majoring in mathematics," by Professor R. V. Churchill, University of Michigan.

Second Session of the Association

Symposium on undergraduate mathematical curricula (continued).

"Mathematics in a liberal education," by Professor E. P. Northrup, University of Chicago.

"The needs of the mathematics major and the graduate student," by Professor E. W. Chittenden, University of Iowa.

"The problem of the returning veteran," by Professor C. V. Newsom, Oberlin College.

Meeting of the Board of Governors

The Board met Saturday afternoon at 3:30 at Hotel Windermere West. Sixteen members of the Board were present, including eight Regional Governors.

The following thirty-five persons were elected to membership on applications duly certified:

Sister Winifred Ahl, A.B. (St. Joseph's Coll., Md.) St. Joseph's Coll., Emmitsburg, Md.

G. B. Banks, Ph.D.(Niagara Univ.) Prof., Physical Sci., Niagara Univ., Niagara University, N. Y.

R. A. Beaumont, Ph.D.(Illinois) Asst. Prof., Univ. of Washington, Seattle, Wash.

Grace L. Bolton, B.S.(N. J. Coll. for Women) Instr., New Jersey Coll. for Women, New Brunswick, N. J.

W. C. Bornmann, A.M.(Columbia) Instr., Georgia School of Tech., Atlanta, Ga.

P. F. Byrd, M.S.(Chicago) 1st Lt., U.S. Army Air Corps.

J. D. Daugherty, A.M.(Pennsylvania) Head of Math. Dept., Eastside High School, Paterson, N. J.

H. P. Fawcett, Ph.D.(Columbia) Prof., Educ., Ohio State Univ., Columbus, Ohio.

Harry Ferguson, B.S.(Boston Univ.) Instr., Tufts Coll., Medford, Mass.

Abe Gelbart, Ph.D.(Mass. Inst. of Tech.) Asst. Prof., Syracuse Univ., Syracuse, N. Y.

T. D. Howe, Jr., B.S. in C.E.(Harvard) T. D. Howe Construction Co., Houston, Tex.

J. B. Jeffries, M.S.(Chicago) Jr. Chemist,

Metallurgical Lab., Univ. of Chicago, Chicago, Ill.

S. A. JENNINGS, Ph.D.(Toronto) Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Can.

R. E. JOHNSON, Ph.D.(Wisconsin) Asst. Prof., Mount Holyoke Coll., South Hadley, Mass.

ROBERTA F. JOHNSON, Ph.D.(Cornell) Asso. Prof., Wilson Coll., Chambersburg, Pa.

H. L. KRALL, Ph.D.(Brown) Asso. Prof., Pennsylvania State Coll., State College, Pa.

SAUL KRAVETZ. Student, Brooklyn Coll., Brooklyn, N. Y.

REV. B. J. KUHN, A.M.(St. Bonaventure) Prof., Siena Coll., Loudonville, N. Y.

W. C. MCDANIEL, Ph.D.(Wisconsin) Asso. Prof., Southern Illinois Normal Univ., Carbondale, Ill.

ROBERT MCLARREN. Managing Editor, Air Age, Inc., 551 Fifth Ave., New York 17, N. Y.

REV. W. J. MILLER, Ph.D.(Harvard) Prof., Woodstock Coll., Woodstock, Md.

E. E. MOISE, A.B.(Tulane) Lt. (j.g.), U.S.N.R.

D. C. MURDOCH, Ph.D.(Toronto) Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Can.

G. M. NAUL, B.Ch.E.(Pratt Inst.) Shift Supervisor, Celanese Corp. of America, Cumberland, Md.

ALBERT NEWHOUSE, Ph.D.(Chicago) Instr., Rice Inst., Houston, Tex.

W. H. NORRIS, JR., A.B. (George Washington) Teacher, Maury High School, Norfolk, Va.

D. O. PATTERSON, Ph.D.(Minnesota) Head of Dept., State Teachers Coll., Valley City, N. D.

C. R. PERISHO, A.M. (Haverford Coll.) Instr., Junior Coll., McCook, Nebr.

BERNARD RASOF, M.S.(Calif. Inst. of Tech.) Research Engr., California Inst. of Tech., Pasadena, Calif.

ERIC REISSNER, Ph.D.(Mass. Inst. of Tech.) Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.

E. H. ROTHE, Ph.D.(Berlin) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

E. J. STULKEN, A.M.(Texas) Statistician, Geophysical Service, Inc., 1311 Republic Bank Bldg., Dallas 1, Tex.

E. G. SWAFFORD, A.M.(Syracuse) Asst. Prof., Park Coll., Parkville, Mo.

L. I. WADE, JR., Ph.D.(Duke) Instr., Duke Univ., Durham, N. C.

D. W. WESTERN, A.M.(Michigan State) Instr., Brown Univ., Providence. R. I.

The Secretary reported the deaths of the following members of the Association:

G. D. Birkhoff, Professor of mathematics, Harvard University. (November 12, 1944)

T. S. Fiske, Professor emeritus of mathematics, Columbia University. (January 10, 1944)

Harris Hancock, Professor emeritus of mathematics, University of Cincinnati. (March 19, 1944)

Edward Helly, Instructor in signal corps training program, Illinois Institute of Technology. (November 28, 1943)

Sister Thomas Marie Maloney, Instructor in mathematics, Trinity College, Washington, D. C. (1943)

J. S. Miller, Professor of mathematics, Emory and Henry College. (March 16, 1944)

Ruth Newlin, Teacher, Junior College and High School, Creston, Iowa. October 20, 1944.

H. L. Rietz, Professor of mathematics, University of Iowa. (December 7, 1943)

T. R. Rosebrugh, Professor of electrical engineering, University of Toronto. (January 24, 1943)

J. A. Shohat, Professor of mathematics, University of Pennsylvania. (October 8, 1944)

D. E. Smith, Professor emeritus of mathematics, Columbia University. (July 29, 1944)

J. R. Wilton, Professor of mathematics, University of Adelaide, Australia. (April 1944)

Clyde Wolfe, Mathematician, Radiation Laboratory, University of California. (March 25, 1944)

The Board elected W. F. Cheney, Jr., as Second Vice-President for a term of two years beginning January 1, 1945.

The Board received a report from the Committee on the Chauvenet Prize consisting of Philip Franklin, Chairman; Saunders Mac Lane, and G. T. Whyburn. The committee recommended that the prize for the years 1941–43 be awarded to R. H. Cameron for his paper, "Some introductory exercises in the manipulation of Fourier transforms," *National Mathematics Magazine*, vol. 15 (1941), pp. 331–356. Acting on this recommendation, the Board voted to award the prize to Professor Cameron.

The Board voted to accept the invitation from McGill University to hold the summer meeting of the Association at Macdonald College of McGill University on Saturday, June 23, 1945. The American Mathematical Society will meet there on Sunday and Monday, June 24–25. These meetings will follow the Canadian Mathematical Congress which will meet at McGill University during the week of June 18–23.

Certain suggestions from the Cooperative Committee on the Teaching of Science were presented to the Board. After some general discussion the suggestions were referred to the Association's Conference Committee on Education for study and subsequent report to the Board.

The resignation of Professor W. D. Cairns from the War Policy Committee was presented, and the Board voted to accept the resignation to be effective January 1, 1945. The Board then elected Professor C. C. MacDuffee to succeed Professor Cairns on this joint committee of the Association and the Society.

On nomination of the Editor-in-Chief the Board elected the following associate editors of the MONTHLY for the year 1945:

E. F. BECKENBACH	OTTO DUNKEL	MARJORIE GROVES
L. M. BLUMENTHAL	H. P. EVANS	B. W. JONES
N. B. CONKWRIGHT	B. F. FINKEL	C. V. NEWSOM
H. S. M. COXETER	J. S. FRAME	P. R. RIDER
W. M. DAVIS	ORRIN FRINK, JR.	MARIE J. WEISS

After President Cairns had retired from the meeting, the Secretary presented a recommendation from the Executive Committee. In view of the long and unique service of Professors Slaught and Cairns to the Association, the committee recommended that the same recognition that had been given to Professor Slaught in 1933 be accorded at this time to Professor Cairns. The Board voted unanimously to recommend to the Association at its annual business meeting that the By-Laws of the Association be suspended and that Professor W. D. Cairns be elected Honorary President for life with the privileges of honorary life membership.

ANNUAL BUSINESS MEETING

The annual business meeting of the Association was held on Sunday morning at 9:30, President Cairns presiding.

On recommendation from the Board of Governors, the Association voted the following amendments to the By-Laws, the effect of these amendments being to abolish institutional memberships in the Association:

ARTICLE II

Section 2 to be eliminated.

In Section 3, after the word "individual" delete the phrase "or institution" and after the word "endorsed" delete the phrase "in the case of individuals."

Sections 3 and 4 to be renumbered as 2 and 3.

ARTICLE VII

In Section 1, delete the word "individual."

In Section 2, delete the word "individual."

Section 3 to be eliminated.

Sections 4, 5 and 6 to be renumbered as 3, 4 and 5.

The results of the election of officers were announced as follows:

President for the term 1945–46: C. C. MacDuffee, University of Wisconsin.

Governors at Large for the term 1945–47: Walter Bartky, University of Chicago; H. S. M. Coxeter, University of Toronto.

An announcement was made of the award by the Board of Governors of the Chauvenet Prize for 1941–43 to R. H. Cameron, Massachusetts Institute of Technology.

The Secretary presented the recommendation from the Board of Governors that the By-Laws be suspended and that W. D. Cairns be made Honorary President for life with the privileges of life membership. The recommendation was adopted by a unanimous rising vote.

W. B. CARVER, Secretary-Treasurer

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held on Saturday, April 15, 1944, at the George Washington University in Washington, D. C. Professor J. H. Taylor, Chairman of the Section, presided at the morning and afternoon sessions.

There were fifty-six persons in attendance, including the following thirty-three members of the Association: O. S. Adams, Harriet W. Allen, M. W. Aylor, T. E. Berry, Archie Blake, Lillian O. Brown, G. R. Clements, Abraham Cohen, J. A. Duerksen, E. J. Finan, B. H. Gere, Michael Goldberg, D. W. Hall, E. H. Hanson, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, J. A. Larrivee, Carol V. McCamman, E. J. McShane, Florence M. Mears, J. F. Milos, T. W. Moore, W. K. Morrill, C. R. Phelps, Grace Shover Quinn, O. J. Ramler, R. E. Root, A. D. Sollins, J. H. Taylor, C. H. Wheeler, III, E. W. Woolard.

At the business meeting the following officers were elected for the coming year: Chairman, C. H. Wheeler, III, University of Richmond; Secretary, W. K. Morrill, Johns Hopkins University; Executive Council, J. H. Taylor, George Washington University, and Michael Goldberg, Bureau of Ordnance; Regional Member of the Board of Governors, D. W. Hall, University of Maryland.

The following program was presented:

1. *Pythagorean representations*, by Lt. (j.g.) C. R. Phelps, U. S. Navy.

2. *A three-space linkage network*, by Michael Goldberg, Bureau of Ordnance, U. S. Navy Department.

It has been proved by A. B. Kempe (1878) that each of the four links of a plane movable quadrilateral linkage can be linked to a common pivot without restricting the movability of the linkage. No three pivots of the resulting configuration need be collinear. Mr. Goldberg demonstrated, with the aid of mechanical models, his three-space analogue of Kempe's theorem. Specifically, he showed that each of the four links of a hinged movable skew quadrilateral linkage can be linked by hinges to a common hinge without restricting the movability of the linkage. No two hinges of the resulting configuration need be parallel or concurrent. No three hinges need have a common perpendicular.

3. *Vertices of plane curves*, by Dr. S. B. Jackson, introduced by the Secretary.

A closed curve of class C'' which is not a circle has two vertices, by virtue of the continuity of the curvature. It was the purpose of this paper to characterize geometrically those curves with exactly two vertices. Let a curve be called normalized if it contains no complete circles, and let a simple closed arc of the curve which is never met again by the curve be called a simple loop. The following facts were established for any normalized curve C having two vertices: (a) the curve C may be divided into two simple arcs; (b) all double points are simple; (c) the curve contains exactly two simple loops, one containing each vertex; (d) none of the regions of the plane bounded by C are bounded always in the same sense except those regions bounded by the loops; (e) at any point of tangency the directed tangents coincide. Slightly different results were obtained for curves which are not normalized. The methods employed were elementary, extensive use being made of the invariance of vertices under direct circular transformations.

4. *Green's functions associated with problems in beam deflection*, by Dr. H. T. Muhly, U. S. Naval Academy, introduced by the Secretary.

The differential equation of the elastic curve of a transversally loaded beam is $y'' = -(1/s)M(x)$, where $s = EI$ is the stiffness coefficient, and $M(x)$ is the bending moment at the point x of the beam. The boundary conditions most frequently associated with this equation are $y(0) = y(L) = 0$, where L is the length of the beam. A widely used formula (a corollary to Castigliano's theorem) for the solution of this problem is $y(x) = (1/s)\int_0^L M(t)m(x, t)dt$, where $m(x, t)$ denotes the bending moment set up by a unit load at x. It was pointed out that this formula has the inherent disadvantage that the computation of $m(x, t)$ involves the solution of an indeterminate problem whenever the beam is statically indeterminate. As an alternative it was suggested that Green's function be used to exhibit the solution. One would then obtain

$$y(x) = \frac{1}{sL}\int_0^x M(t)t(L - x)dt + \frac{1}{sL}\int_x^L M(t)x(L - t)dt.$$

It was shown that this representation enabled one to derive in one step the familiar "three moment theorem" of Clapeyron. The generality of the method was brought out by applying it to the cases in which the beam carried an axial compression or tension in addition to the transverse load. The Green's functions for these cases were given, and the solutions obtained thereby were used to derive generalizations of Clapeyron's theorem to these more difficult cases.

5. *Mathematics as she are taught*, by Dr. L. B. Tuckerman, introduced by the Secretary.

The speaker pointed out many erroneous and ambiguous details in mathematics texts and other publications. He urged that all teachers make an effort to be more precise in their teaching.

W. K. MORRILL, *Secretary*

The Editor-in-Chief wishes to express his indebtedness to the following persons who have served as referees of papers during 1944.

E. F. Beckenbach, L. M. Blumenthal, W. C. Brenke, R. W. Brink, B. H. Brown, H. E. Buchanan, W. E. Buker, R. S. Burington, J. W. Campbell, W. B. Carver, W. B. Caton, N. B. Conkwright, A. H. Copeland, H. S. M. Coxeter, C. C. Craig, John DeCicco, H. J. Ettlinger, J. S. Frame, M. G. Gaba, M. R. Hestenes, T. H. Hildebrandt, Dunham Jackson, W. C. Krathwohl, Mayme I. Logsdon, C. C. MacDuffee, G. M. Merriman, E. J. Moulton, J. R. Musselman, Rufus Oldenburger, Isaac Opatowski, Haim Reingold, P. R. Rider, M. A. Sadowsky, I. M. Sheffer, E. B. Stouffer, T. Y. Thomas, T. L. Wade, Marie J. Weiss, and F. E. Wood.

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA, Cedar Rapids, April 14, 1945
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, Brooklyn, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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